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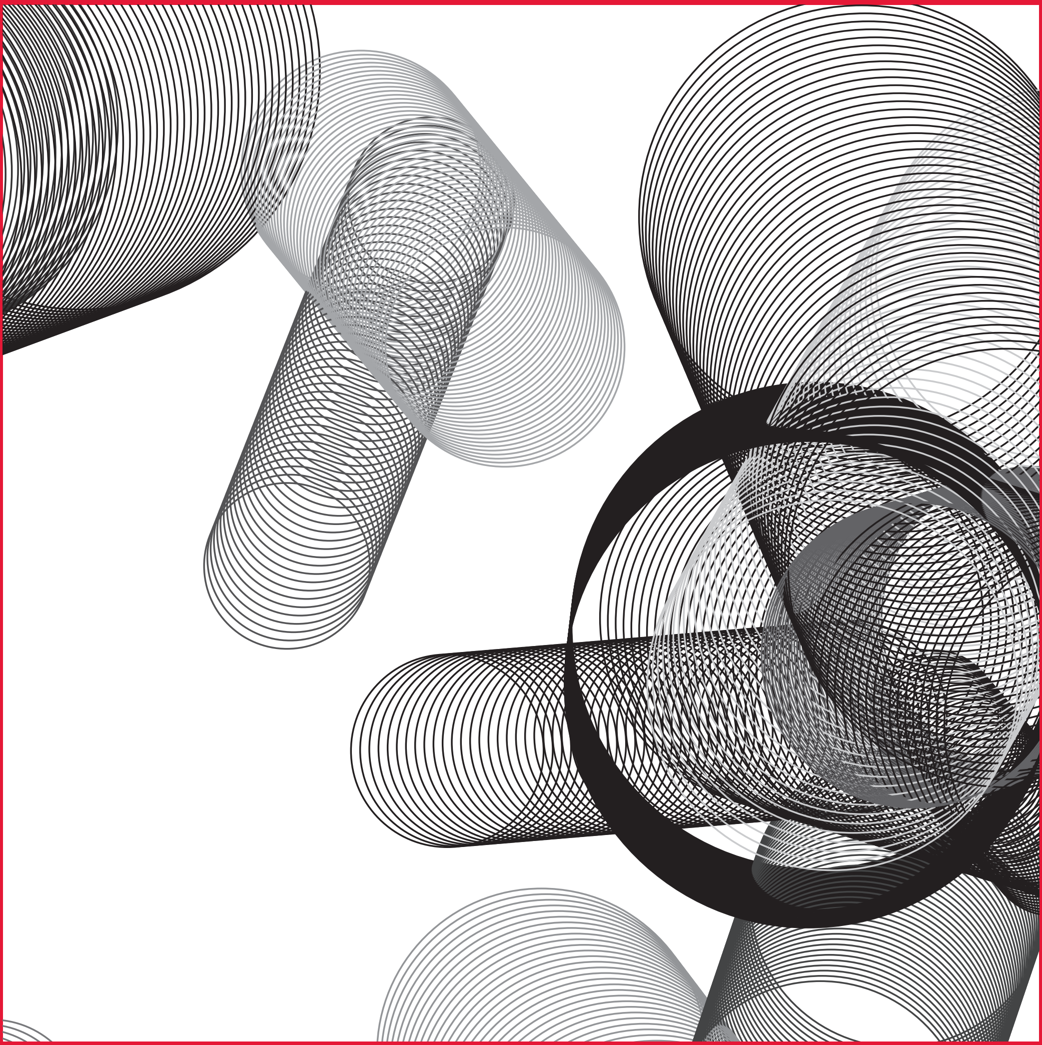

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- Self-Tiling Tile Sets
- How Remarkable Is This Polyhedron?
- Reflexive Polygons and Mirror Symmetry

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LETTER FROM THE EDITOR

In the fine art of tiling, Lee Sallows is an acknowledged master. In this issue he generalizes the idea of a *rep-tile,* a shape that can tile a larger version of itself. On the facing page you see an example of a *self-tiling set* of four tiles which, as a set, tile larger versions of each of themselves. Do you like the circularity? Turn the page to see where it leads.

Hans Fetter's article is about a certain lattice polyhedron—that is, a polyhedron in $\mathbb{R}^3$ whose vertices have integer coordinates. Of its many special properties I'll mention just one: Its dual polyhedron, or more precisely, its *polar set,* is also a lattice polyhedron. And not just any lattice polyhedron! Read on for the full story.

Charles Doran and Ursula Whitcher also consider lattice polyhedra, and especially, lattice polyhedra whose polar sets are also lattice polyhedra. Actually, they start more gently, in $\mathbb{R}^2$, with lattice polygons. In $\mathbb{R}^2$ these objects are called *reflexive polygons,* and in higher dimesions, *reflexive polytopes.* (Unfortunately for Fetter, they use a less flexible definition of "polar set" that leaves Fetter's polyhedron out in the cold.) Their article does not finish as gently. Starting from reflexive polytopes they show how to construct Calabi-Yau manifolds, which play a fundamental role in string theory. This paper won't tell you everything about Calabi-Yau manifolds, but it will tell you why they appear in "mirror pairs," which may be as good a start as any toward mastering the theory of everything.

Our Notes authors address some classical problems. Daniel Daners offers a proof of the famous formula for $\pi^2/6$, requiring little more than integration by parts. Michael Sheard evaluates another limit, which can also be used to determine π, at least in places where π is 3. Marshall Ash finds the limits of the limit comparison test, and Sunil Chebolu tells a story about the divisors of 24. Finally, Will Murray gives a name to *Möbius polynomials* and uses them to prove Euler's totient theorem. To close the loop, our Proofs Without Words all come back to tiling.

In this issue we again recognize our excellent referees. One can never thank them too often. Take the time to browse through the list, so that when you see one of these generous and hard-working people, you can buy him or her a polar polytope. We depend always on our authors; we continue to receive more excellent submissions than we can accept, and we value the talents and efforts of all contributors, regardless of outcome. I want to extend thanks personally to Swarthmore and Bryn Mawr Colleges for their help and community, and to acknowledge the efforts of the Associate Editors, the MAA staff members, the compositors, and the many others who do so much to make this MAGAZINE possible.

Walter Stromquist, Editor

ARTICLES

On Self-Tiling Tile Sets

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> The first telephone was invented in 1853. However, the device never really caught on until the invention of the *second* telephone some time later.
>
> —Spike Milligan [author's paraphrase]

An animal familiar to polyform specialists is the *rep-tile*, or self-replicating tile, first introduced by Solomon Golomb [**1**]. It is a planar shape that can be tiled with smaller replicas of itself. Hitherto, rep-tiles have usually been regarded as singular creatures exhibiting few links with the rest of the animal kingdom. Here I suggest that with only a small change in perspective they can be identified as close cousins of a previously unexplored object that I call a *self-tiling tile set*.

By a self-tiling tile set of order n, I mean a set of n distinct (non-similar) planar shapes, each of which can be tiled with smaller replicas of the complete set of n shapes. We require that the scaling factor be the same for each piece. FIGURE 1 shows an example for $n = 4$ formed of hexominoes. Note that some pieces are flipped.

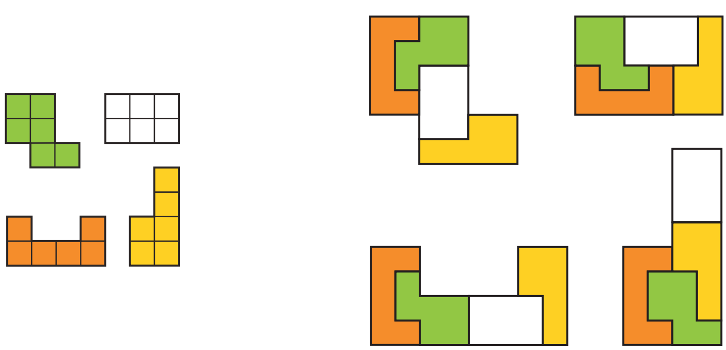

Figure 1 A self-tiling tile set of order 4 using hexominoes

Here the four hexomino-shaped areas at right are each tiled by the four similar hexominoes at left. Or, to look at the same property through the other end of the telescope, the four shapes shown, small or large, can each be dissected into four distinct pieces that are smaller duplicates of the same four figures. Clearly, the twin actions of forming still larger and larger copies (known as *inflation*), or still smaller and smaller dissections (*deflation*), can be repeated indefinitely. Here we consider sets using flat pieces only, but the extension to 3-D pieces is obvious.

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Note that, since the n members of any such set are each tiled by the same n pieces, their areas are necessarily equal. Also, since the area of any compound copy formed by combining the n pieces is n times that of its smaller version, the increase in *scale* is $\sqrt{n}$. For example, in FIGURE 1 the area increase is 4 times, indicating a scale factor of $\sqrt{4} = 2$, meaning that larger pieces are twice the size of smaller. Hence, if a set is to use n *polyforms*, which have integral sizes, then the compound shapes produced by combining n of them will also have integral sizes, showing that $\sqrt{n}$ must be an integer. A self-tiling tile set of polyforms using a non-square number of pieces is thus impossible, so that beyond $n = 4$ the next possibility becomes $n = 9$.

How do we go about finding such a set? Although straightforward in principle, a search by computer turns out to be quite demanding. In examining the hexominoes, for example, of which there are 35 distinct shapes, I began with a program that identifies every possible tiling of a *bighex* (a doubled-in-size-hexomino-shaped area) by four distinct hexominoes. The latter are each represented by a distinct integer, a *hexnum*, so that the program's output is in the form of a list of *quads*, each consisting of 4 hexnums representing a distinct tiling of the bighex. Repeating this process for every bighex in turn, I ended up with 35 lists of quads. Many of these quads occur in more than one of the 35 sets, since the same 4 pieces often tile different bighexes. Taking each quad in turn, a second program then constructed a list of those bighexes it tiles (the latter again represented by their corresponding hexnums), followed by a search for any quad, all 4 of whose members appear among its associated list of bighexnums. Only one such quad was found; it is the one represented by FIGURE 1. Alas, sets for which $n > 4$, involving still larger bighexes, were beyond the scope of the program and thus remain unexamined. My grateful thanks are due to polyforms expert Pat Hamlyn, who was kind enough to supply me with the software used in the researches here described.

Stimulated by this single find, I turned next to the heptominoes, of which there are 108. Patiently entering data to the program so as to define the shapes to be tiled was a lengthy task, requiring some two days' determined effort. It came as quite a blow when the program discovered no solutions whatever. Worse yet, subsequent experiments yielded no further solutions for any polyominoes smaller than octominoes. There are 369 octominoes, a forbidding total in view of the work their examination entails. When at length I did launch into an exploration of the octominoes, I found myself surveying a staggeringly different world.

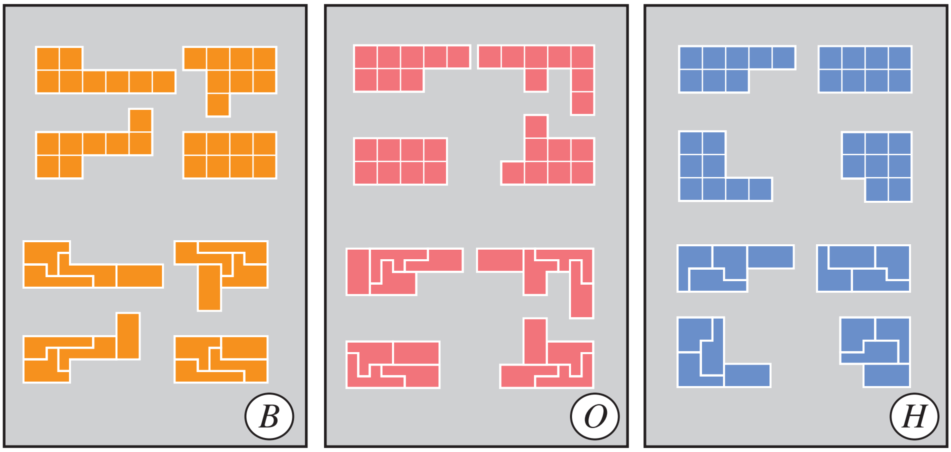


Figure 2 Three of the seven self-tiling tile sets using 4 octominoes

The octomino network for $n = 4$

To begin with, there exist as many as *seven* distinct self-tiling tile sets using four octominoes. Three of them are shown in FIGURE 2, labeled B, O, and H; the remaining four can be found in FIGURE 8. These were pleasing finds, but on running an eye over the computer output data, I soon noticed signs of a phenomenon already foreseen as a possibility from the start, namely, the existence of loops, which is to say, of closed chains of sets, each of which tiles its successor. After making a few changes to the program, I settled down to study the loops in detail. FIGURE 3, for example, records one of the length-2 loops found, which is to say, a pair of *co-* or *mutually*-tiling sets. The two sets of four pieces shown are distinct, although one octomino, the rectangle, is common to both. The fact that the same piece occurs in both sets is perhaps a pity, but acceptable so long as our requirement is merely that *sets* be distinct.

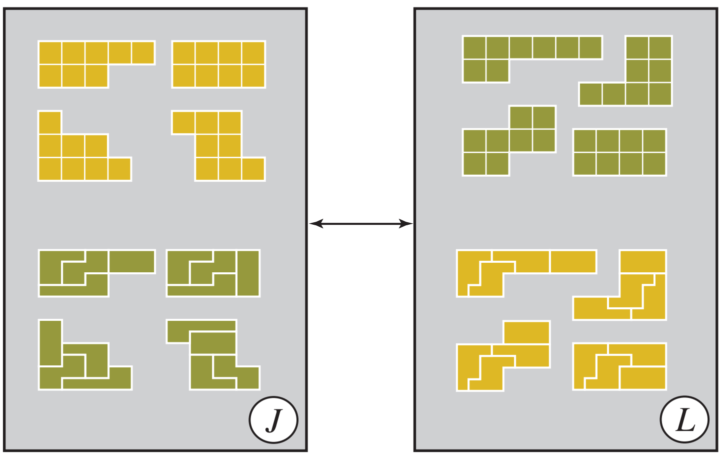
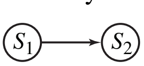
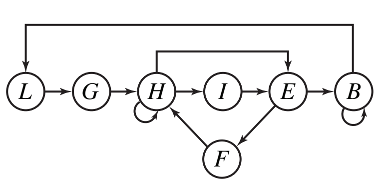


Figure 3 A pair of mutually tiling sets or a closed loop of length 2

Curiously, the same rectangular octomino appears in every one of the sets met with below. In FIGURE 3, the four green octominoes tile (a larger replica of) the yellow, that in turn tile (a larger replica of) the green. From here on it will be convenient to take the phrase "a larger replica of" as understood, and to speak simply of one set tiling another. Likewise, to save space, tiled shapes in the sets pictured are also shown non-enlarged. In the figures to follow, an arrow pointing from set S_1 to set S_2 indicates that S_1 tiles S_2, a relation we can represent symbolically by a directed graph thus:



The need for some such notation made itself felt, as the loops emerging from the data became not merely longer, but even intertwined with each other in complicated ways. For example, FIGURES 4 and 5 show two of the loops found, one of length 3 and one of length 6, respectively. But sets E and H appear in both loops, while sets B and H are among the self-tilers in FIGURE 2, a tangle that is nevertheless neatly captured by the following graph:



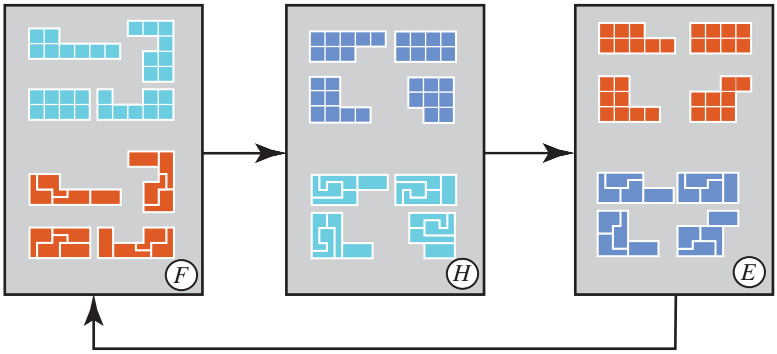


Figure 4 A loop of length 3

Armed with this simple device, initial attempts to map out the complete octomino network soon began to reveal an extraordinary complexity. It was a complexity that quite overwhelmed my elementary methods. Thus far, loops had been traced by hand. Before long it became clear that such an approach could never identify them all. FIGURE 7, for example, itself a veritable rat's nest of interconnections, records a mere fragment of the whole. It is a graph of those loops associated with a restricted subset of the octominoes, namely those small enough to fit within a 12×6 rectangle. Even so, FIGURE 7 is of illustrative use. Observe that its lettered nodes correspond to the lettered sets of same colour appearing in other figures. Inspection will reveal 9 cases of lines with double-headed arrows, indicating 9 loops of length 2. Interested readers may enjoy verifying its structure through tracing the remaining loops, several more of which can be found for all lengths up to a maximum of 8. To this end, for completeness, the six sets A, C, D, K, M, and N, which do not appear in other figures, are included in FIGURE 6. The task of discovering how the pieces of one set fit together so as to tile those of another must then be arrived at by trial and error. Note that while set

Figure 5 A loop of length 6

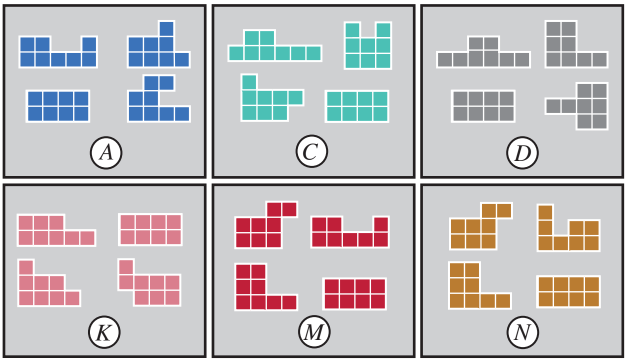


Figure 6 Six remaining sets that (together with those shown in the other figures) enable a complete verification of FIGURE 7

O in FIGURE 2 tiles sets A, B, D, F, and L, as well as tiling itself, it does not form part of any other loop, for which reason it is not included in FIGURE 7.

Meanwhile, the problem of unraveling the complete network had defeated my every effort. Small wonder then that I sought outside help. In the aftermath, warm thanks are due to my ex-colleague Henk Schotel, formerly a cognitive scientist at the Radboud University in The Netherlands, whose application of Frank Meyer's Java program [**2**] implementing Tarjan's well-known graph-searching algorithm [**3, 4**], provided a perfect tool for the job. Almost all the credit for securing a complete description of the network belongs to Henk Schotel, without whose generous help and expertise this account would be much the poorer.

Nevertheless, it is a matter of regret that the network brought to light by Schotel is simply too large, and its interconnections too dense to be captured in a legible

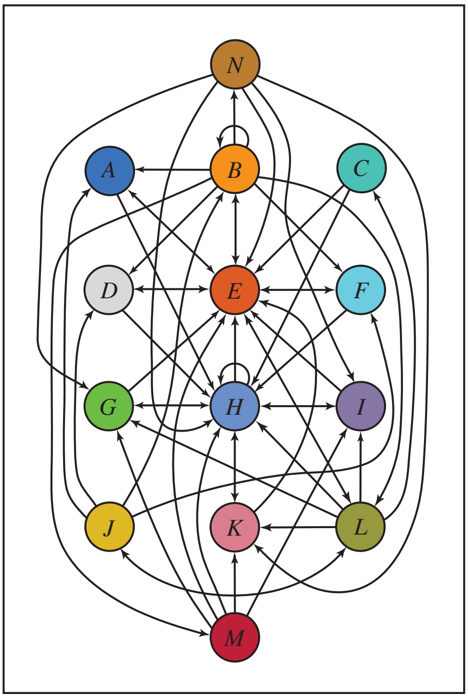


Figure 7 A small fragment of the complete octomino network

graph, containing as it does 62 nodes joined by 461 links. I have already mentioned that this includes seven self-tilers. Beyond these, however, there appears an astounding plethora of loops, ranging in length from 1 up to 14. TABLE 1 records their precise frequencies, several of them in the hundreds of thousands. The total number of loops found is thus not far short of one and a half million. Alas, even to provide the very minimum of information necessary to enable readers to reconstruct the network for themselves would again require overmuch space here. For example, among other data required, the adjacency matrix involved is of size 62 × 62. In consequence, full details together with explanatory notes have been made available on the author's website, http://LeeSallows.com.

TABLE 1: Loop lengths and their frequency of occurrence in the octomino network

Loop length	Number of loops
1	7
2	31
3	162
4	807
5	3,330
6	11,413
7	32,683
8	78,384
9	158,040
10	260,408
11	334,896
12	316,800
13	186,240
14	46,080
	1,429,281

By means of this, polyform enthusiasts curious to explore the octomino labyrinth for themselves can be sure of encountering some arresting structures. One such is what I call the 'sunburst', so-called because its corresponding digraph consists of a central node (set E in FIGURE 5) surrounded by 14 satellite nodes to which it is linked by 14 'rays' of double-headed arrows. Or in other words, set E is the common member of an astonishing 14 separate loops of length 2. Once again, limitations of space make it impractical to depict all 14 of these sets here. Set E is in fact by far the 'busiest' node in the network, occurring as it does in over 99.9% of all loops. Still more striking is the 'quintet', an amazing family of five sets pictured in FIGURE 8. Every n-tuple of sets forms a bidirectional loop of length n. That is, every set is a self-tiler, every pair of sets is a loop

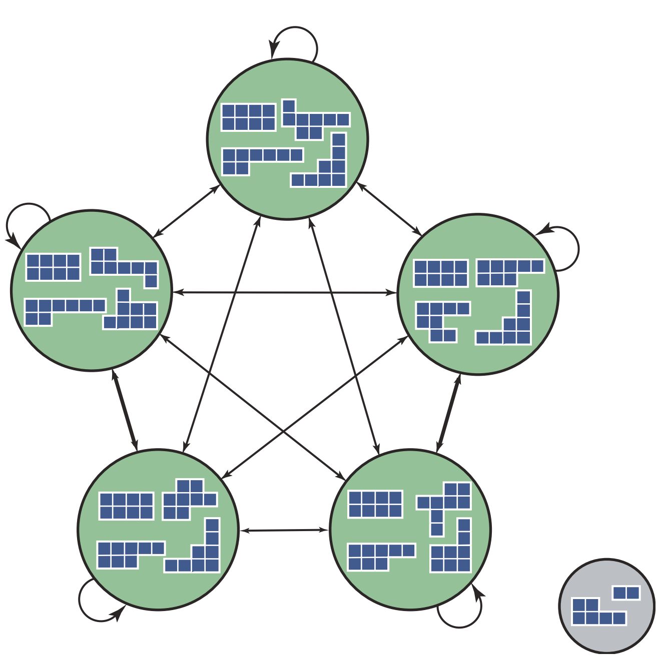


Figure 8 "The Quintet": Every set is a self-tiler; every pair of sets tile each other.

of length 2, every triad of sets is a loop of length 3, and so on. As a result, the 4 pieces in any one of the 5 sets will assemble to yield a twice-sized copy of any chosen piece. It is easy to explain how this adaptability comes about. Examination will confirm that every one of the 11 distinct octominoes appearing can be formed by distinct juxtapositions of just two component shapes: the hexomino and domino shown below right in FIGURE 8. Less obvious is that the three non-rectangle pieces in each set will themselves combine to produce an enlarged copy of this hexomino. But this means that the latter can then be juxtaposed with the remaining rectangle (= enlarged domino) so as to yield larger duplicates of any of the original 11 pieces. Incidentally, with just three exceptions, the rectangular octomino is found in all 62 sets of the complete octomino network.

A point worth noting here is that, although structures such as the sunburst and quintet are 'there' in the network, they still have to be sought for and identified amid the tightly interlacing reticulation. The process of finding them is thus not unlike that of the sculptor, whose task is to chip away stone obscuring the image hidden within the block. Recall too that in practice we are not looking at a real digraph such as FIGURE 7, but at computer output in the shape of lists of numbers corresponding to such a graph. There is thus plenty of scope for serious detective work in the process of tracking down patterns of interest concealed within the tracery. So much then for a brief account of the reflexive tiling properties of the octominoes for the case $n = 4$.

Beyond polyforms

Thus far we have made no more than an initial foray into an as yet largely unexplored field. Doubtless future explorers will extend these researches to still other and larger polyforms.

However, the pieces used in a self-tiling tile set do not have to be polyforms; they may be planar shapes of any kind, in which case n, the number of pieces in the set, need not be restricted to *square* numbers. But in that case, what of the *smallest* possible self-tiling tile set, which is of order 2, or one using just two pieces? The question thus prompted may be stated as follows:

Do there exist two non-similar planar figures, A and B, such that each can be dissected into two smaller figures, A′ and B′, where A′ is similar to A and B′ is similar to B?

It is a problem that occupied me for weeks before finally hitting on a whole class of solutions using two triangles. We shall look at them below. In the meantime, the

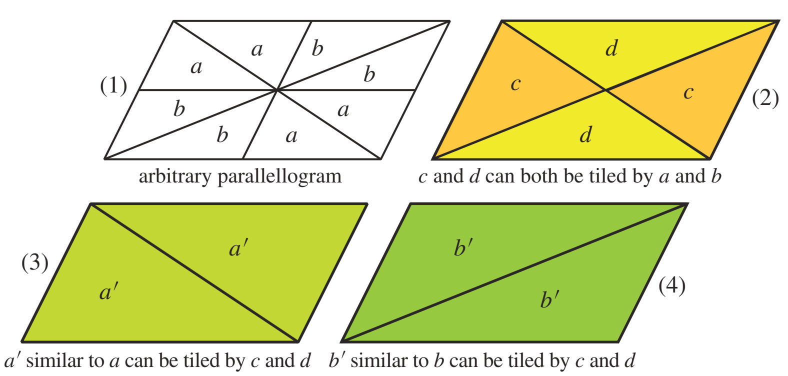


Figure 9 The parallelogram method for producing a pair of co-tiling sets composed of two triangles

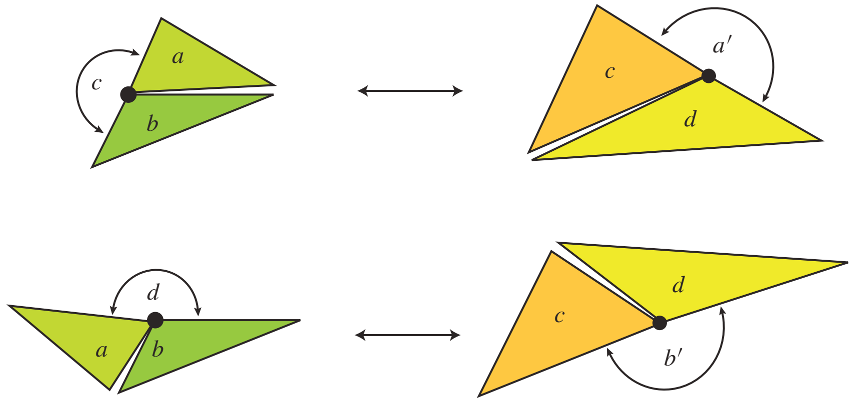


Figure 10 Above: triangles a and b tile c, while c and d tile a', which is similar to a. Below: triangles a and b tile d, while c and d tile b', which is similar to b.

finding is more informatively introduced by considering first a solution to a related problem, which is that of discovering a *pair* of mutually-tiling sets, each composed of two pieces.

Consider FIGURE 9(1), which depicts an arbitrary parallelogram divided into 8 triangles by the four bisectors shown. The triangles are of two distinct shapes, labeled a and b. FIGURE 9(2) identifies two distinctly shaped regions c and d, each of which corresponds to a distinct union of a and b. That is, both c and d can be tiled with the triangles a and b. In similar fashion, triangle a' in FIGURE 9(3) is a union of c and d, just as triangle b' in FIGURE 9(4) is a different union of c and d, so that each of the triangles a' and b' can be tiled by the triangles c and d. But a' is similar to a, and b' is similar to b, which completes a loop of length 2. In short, triangles a and b tile both c and d, which in turn tile (larger copies of) a and b. Note that since we may start with any parallelogram, the variety of triangle shapes in the sets resulting can be varied continuously. Less obvious is that the dissections of triangles a', b', c, and d that result are all *hinged* dissections, as illustrated in FIGURE 10. Here I have omitted relative angles and side lengths, which are easily derivable from the initial parallelogram chosen.

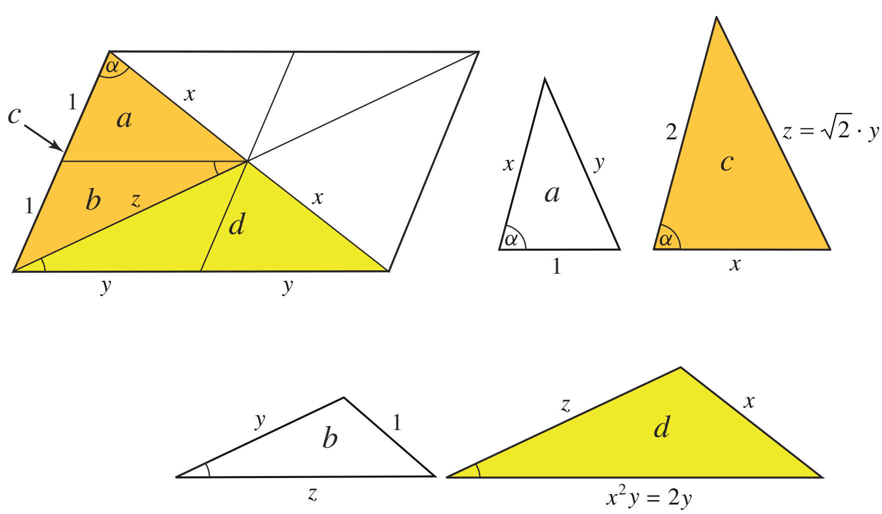


Figure 11 If triangles a and b are similar to c and d, then x must equal $\sqrt{2}$.

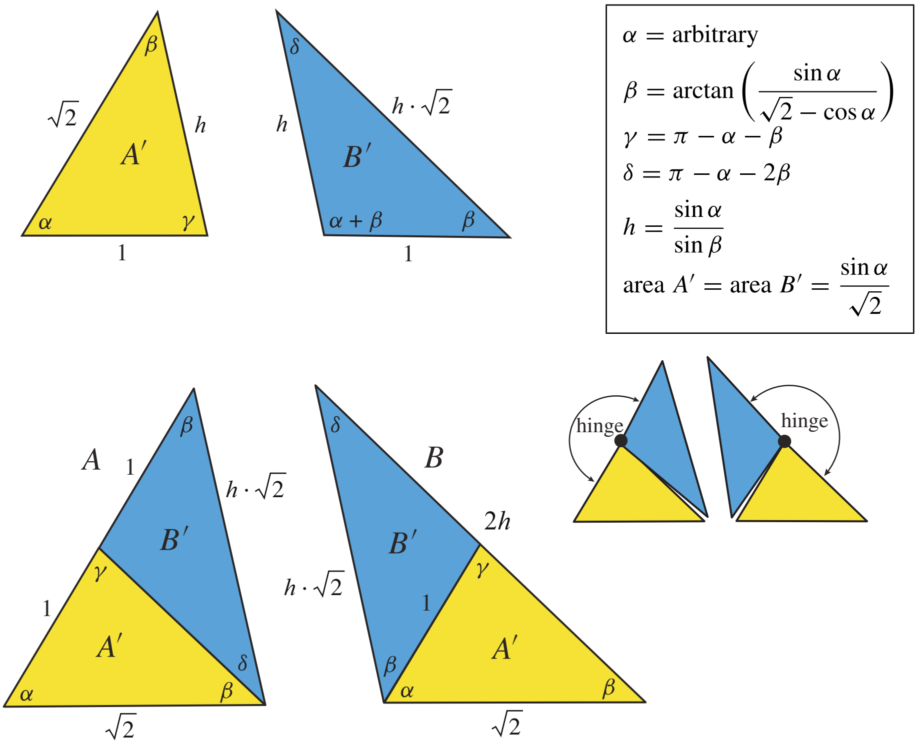


Figure 12 A class of self-tiling tile sets of order 2: A and B are similar to A' and B', respectively.

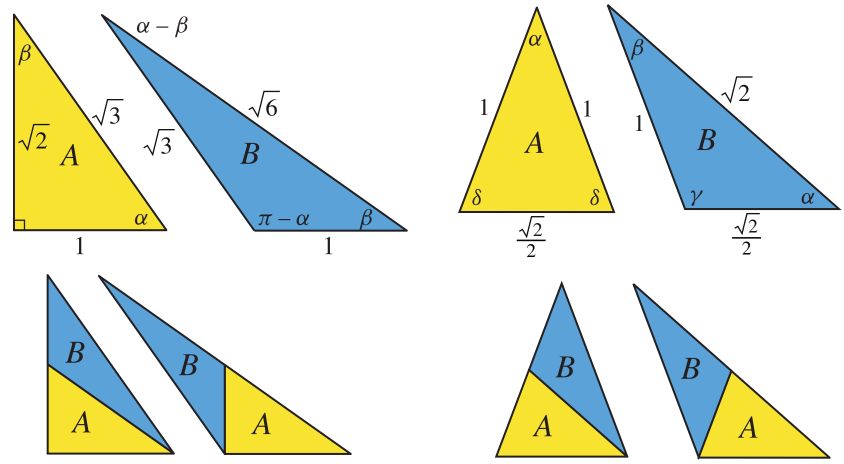
This completes a look at the "parallelogram method" for producing a pair of co-tilers comprising two triangles in each set. A couple of special cases to consider are when the parallelogram employed is either a rectangle or a square, with the result that triangles a and b become congruent. A more interesting case occurs when the parallelogram chosen is such as to make triangles c and d *similar* to a and b, respectively. The outcome is then that triangles a and b are able to tile larger versions of *themselves*, which is exactly the property at the focus of the question above.

A family of order-2 sets

Under what conditions will triangles a and b be similar to c and d, respectively? FIGURE 11 reproduces the parallelogram of FIGURE 9(1). Labels identify edge lengths in the various triangles. We can assume without losing generality that the short side of triangle a is of length 1. Suppose now a and c are similar; each side of a thus has its corresponding side in c. The angle α appears in both, and both share a side of length x. But triangle c is larger than a, so that the side of length x in one cannot correspond to the side of length x in the other. In light of these points, to the right, triangles a and c have been reproduced (rotated and reflected), but now oriented so that their common angle, α, appears below left in both, with corresponding sides now in corresponding positions. The difference in base lengths shows that the scale factor must be x, so that on comparing the left hand edges of a and c, we find $x^2 = 2$. Hence in triangle c, $z = x \cdot y = \sqrt{2} \cdot y$. Analogously, in FIGURE 11, triangles b and d are likewise oriented, their common angle and resulting edge lengths confirming that, like a and c, triangles b and d are similar exactly when $x = \sqrt{2}$.

$$\alpha = \arcsin\left(\frac{\sqrt{2}}{\sqrt{3}}\right) = \arccos\frac{1}{\sqrt{3}}$$
$$\beta = \arcsin\left(\frac{1}{\sqrt{3}}\right) = \arccos\frac{\sqrt{2}}{\sqrt{3}}$$

$$\alpha = \arccos\left(\frac{3}{4}\right) \qquad \gamma = \frac{\pi+\alpha}{2}$$
$$\beta = \delta - \alpha \qquad \delta = \frac{\pi-\alpha}{2}$$



Triangle A is right-angled   Triangle A is isosceles

Figure 13 Two instances of the triangle pair in FIGURE 12 when one of them is right-angled (left) or isosceles (right)

In summary, given any parallelogram in which one diagonal is equal to $\sqrt{2}$ times the length of one side, then the same ratio obtains for the other diagonal, and the four triangles into which the two diagonals divide the parallelogram are similar to the two triangles into which one diagonal divides it, together with the two triangles into which the other diagonal divides it. In consequence, the two triangles corresponding to a and b (or c and d) in FIGURES 9, 10, or 11 will then furnish a pair of figures possessing the properties of a self-tiling tile set: a and b will together tile either c or d, which is to say, will tile larger versions of a or b themselves.

The properties of any such pair of triangles are summarized in FIGURE 12. Again, a couple of interesting cases occur when triangle A is either isosceles or right-angled, as seen in FIGURE 13. Whether there exist any order-2 sets different to those captured

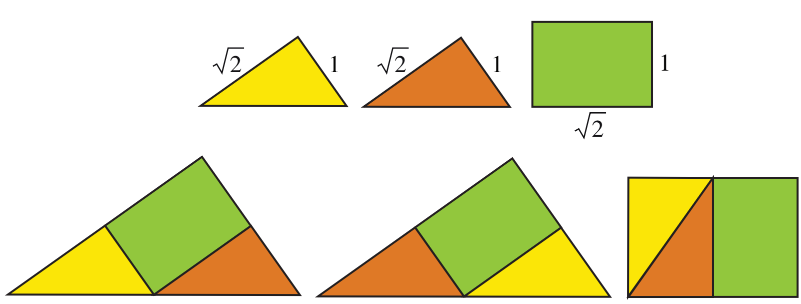


Figure 14 Frank Tinkelenberg's near solution using pieces of distinct area (and shape). The small triangles become doubled in size, while the larger rectangle is $\sqrt{2}$ times larger than the smaller. The latter can be replaced by any $1 \times \sqrt{2}$ parallelogram with appropriately modified triangles.

in the class of solutions here identified remains an unanswered question. Note that triangles A and B become congruent when $\alpha = 45°$.

Before closing, a point worth mentioning is the striking similarity to be found in comparing the triangle pairs considered above with another famous triangle duo known as the Robinson triangles. I refer here to the two “golden” triangles, first identified by Raphael Robinson [**7**], that result from bisecting the kites and darts familiar to us from Penrose tilings. (The triangles are also described by Penrose [**6**, pp. 32–37] and by Grünbaum and Shephard [**5**, Sec. 10.3].) Is there any significance to be read into the obvious correspondences between the two pairs? Alas, no. A simple procedure by means of which, given any triangle, a second can be constructed to result in a pair exhibiting analogous properties is easy to produce. Or in other words, in this respect the Robinson pair is merely one instance of an infinite family of such pairs, all of which are equally “significant.”

Finally, I should like to say a word about the requirement included in the definition of a self-tiling tile set, the effect of which is to guarantee that the n enlarged (compound) copies of the n pieces are equal in size. It is the stipulation that *the scaling factor be the same for each piece.* For only thus does it follow that the n pieces must be of same area, a property that is assumed throughout the discussion above.

Nevertheless, although natural enough, such a demand is not essential. Consider an *unrestricted* self-tiling tile set, in which pieces may be of different sizes because the *scale* of the compound copies formed is permitted to differ from piece to piece. FIGURE 14 shows a near miss at such a solution using three pieces, due to Frank Tinkelenberg. Alas, two pieces are identical, despite which imperfection the use of pieces of distinct area is nicely demonstrated. Perhaps this example may encourage some readers to give thought to the unsolved problem of identifying a flawless self-tiling tile set of order 3, restricted or otherwise.

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Summary A novel species of self-similar tilings is introduced. Results divide naturally into two categories, involving either polyforms or non-polyforms. Some eye-catching findings are presented in what promises to be a rich field of research. The discovery of a class of triangle pairs showing arresting properties will undoubtedly surprise and appeal to many, not least those readers already conversant with rep-tiles.

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A Polyhedron Full of Surprises

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Let's become acquainted with our special polyhedron right away. In FIGURE 1 we see a convex polyhedron having 20 faces, 38 edges, and 20 vertices. It is a centrally symmetric polyhedron embedded in $\mathbb{R}^3$ with its center at the origin.

Any polyhedron with 20 faces can be called an icosahedron. But to avoid any possible confusion between it and other icosahedra, in particular the regular icosahedron, we shall refer to it as *Kirkman's Icosahedron* in honor of the British mathematician Thomas Penyngton Kirkman (1806–1895). It does not appear in his work, but as we will see, many of its properties are related to those he studied. Consult Biggs [**2**] for more details about his life and work, or read some of his own publications on polyhedra [**9, 10**].

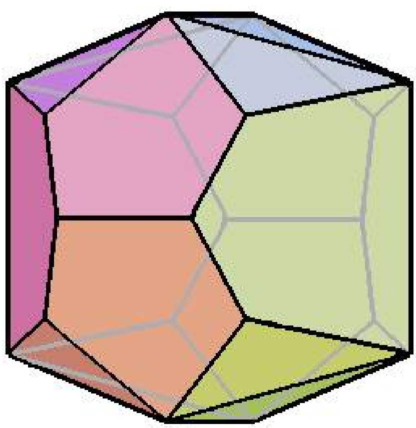

Figure 1 Kirkman's Icosahedron.

This polyhedron has many amazing properties. Let's have a look at them.

First set of surprises

We were able to construct it so that all twenty of its vertices have integer coordinates:

$$(\pm 9,\ \pm 6,\ \pm 6)$$
$$(\pm 12,\ \pm 4,\ 0)$$
$$(0,\ \pm 12,\ \pm 8)$$
$$(\pm 6,\ 0,\ \pm 12)$$

They are also indicated in FIGURE 2.

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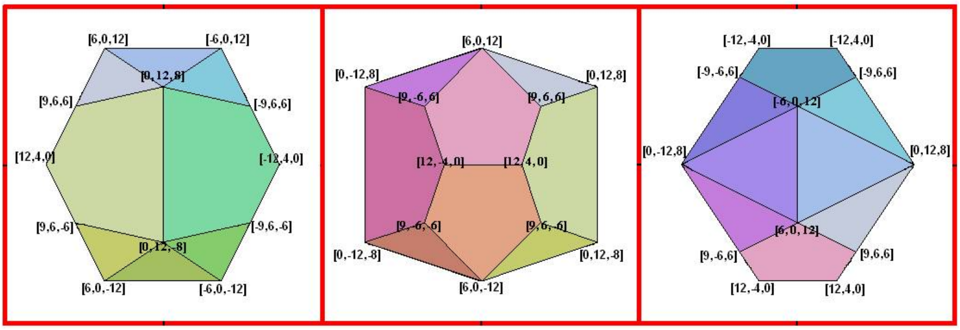

Figure 2 Three views of Kirkman's Icosahedron.

As for the lengths of its edges, it is easy to prove that they are all integers as well and their values are restricted to the numbers in the set

$$\{7, 8, 9, 11, 12, 14, 16\}.$$

A Schlegel diagram for Kirkman's Icosahedron, a projection of it onto the plane, is helpful because it allows us to display these lengths more clearly (see FIGURE 3).

Now, what about the volume of Kirkman's Icosahedron?

To compute it let us join the origin to all the vertices and edges of the object, splitting the polyhedron up into a collection of pyramids. The volume is then the sum of the volumes of these constituent pyramids. There are only four different kinds of pyramids: two with pentagonal bases and two with triangular bases. TABLE 1 is a summary of our findings. Each pyramid's volume is an integer!

From these results we finally get

$$4 \cdot 384 + 4 \cdot 576 + 4 \cdot 288 + 8 \cdot 192 = 6528$$

for the volume of Kirkman's Icosahedron.

Now, let us turn our attention to a completely different topic. What happens when we consider the traveling salesman problem on the surface of our polyhedron?

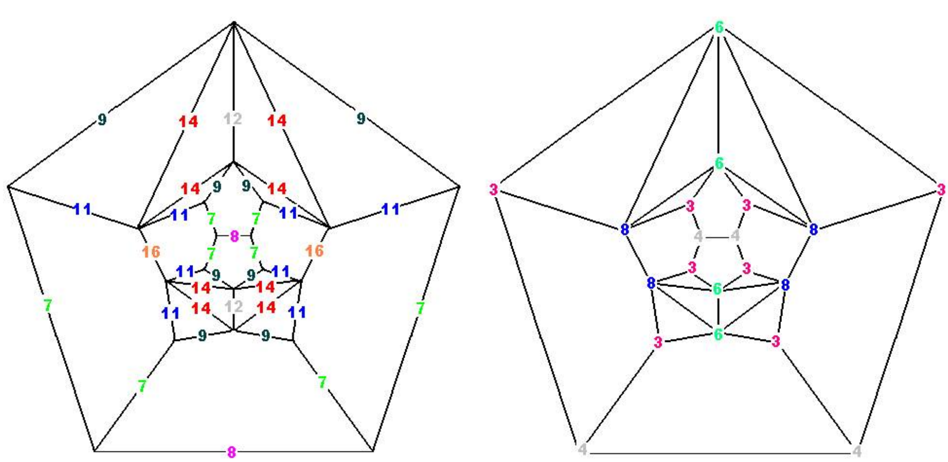

Figure 3 Schlegel diagram for Kirkman's Icosahedron, showing the lengths of the edges and corresponding weights for the vertices.

TABLE 1: Volumes.

	B	h	Volume
pentagon I	$48\sqrt{5}$	$24/\sqrt{5}$	384
pentagon II	$48\sqrt{13}$	$36/\sqrt{13}$	576
triangle I	$24\sqrt{10}$	$36/\sqrt{10}$	288
triangle II	$12\sqrt{17}$	$48/\sqrt{17}$	192

Second surprise

The study of cycles on polyhedra can be traced back to Kirkman's work. In [**9**] he considered the following question: Given the graph of a polyhedron, can we always find a cycle that passes through each vertex once and only once? The traveling salesman problem in addition takes the intermediate distances between the vertices into consideration, and asks to find a cycle of shortest possible length. All the information we need in order to solve this problem can be gathered from the Schlegel diagram in FIGURE 3. First we determine the number of possible closed tours (Hamiltonian cycles) for Kirkman's Icosahedron. With the help of a computer program, we find that there are precisely 206 of them. Next we need to solve the question: Which of these has the minimum total length? We could examine them one by one or use the following shortcut.

Suppose that, instead of using the lengths for the edges in our computations, we decide to attach certain weights $w(u)$ to the vertices u that we encounter in the Schlegel diagram. But how, exactly, are we supposed to assign these weights to the vertices? We choose them so that the length $c(u, v)$ of each edge (u, v) can alternately be computed as $c(u, v) = w(u) + w(v)$. It is a simple exercise to see that this can be done only in one way (see the right image of FIGURE 3).

Since an arbitrary Hamiltonian cycle H has to include every vertex of S exactly once, its overall length is just twice the sum of the weights of all the vertices:

$$\sum_{u,v\in H} c(u, v) = 2\sum_{u} w(u) = 192.$$

So all 206 cycles have the same total length!

In other words, for Kirkman's Icosahedron we get what is known as a *constant* traveling salesman problem (for additional information on this problem, consult [**8**]).

Next, let us briefly discuss a very interesting notion in the theory of polyhedra: duality. What can be said about the dual of Kirkman's Icosahedron?

Third surprise

We need to concentrate on two important concepts for polyhedra: isomorphism and duality, or *syntypicism* and *polar-syntypicism*, as they were referred to in one of Kirkman's very entertaining papers on polyhedra (see [**10**]). We consider these first from a combinatorial point of view.

When, in a typical book on polyhedra (take for instance [**7**]) an author says:

- a truncated square pyramid and a parallepiped are *isomorphic*,
- or a distorted convex prism and a squashed convex dipyramid are *dual* to each other,

what exactly does he or she mean by that? Grünbaum and Shephard [**4, 5**] and also Ashley et al. [**1**] suggest the following definitions:

- Two convex polyhedra P_1 and P_2 are *isomorphic* if there is a one-to-one correspondence between the family of vertices and faces of P_1 and the family of vertices and faces of P_2 which preserves inclusion.
- Two convex polyhedra P and P^* are said to be *duals* of each other if one can establish a one-to-one correspondence between the family of vertices and faces of P and the family of faces and vertices of P^* which reverses inclusion.

Note that these notions belong to the combinatorial or topological theory of convex polyhedra, because they both ignore any geometrical aspects.

Although we could easily find the dual of Kirkman's Icosahedron directly from the Schlegel diagram, we prefer to employ a more constructive approach involving the notion of polarity. It is an appropriate tool that will allow us to visualize a pair of dual polyhedra as an arrangement of interpenetrating solids. A familiar example involving the regular dodecahedron and its dual, the regular icosahedron, is shown in FIGURE 4. For additional pairs, see also [**7, 16**].

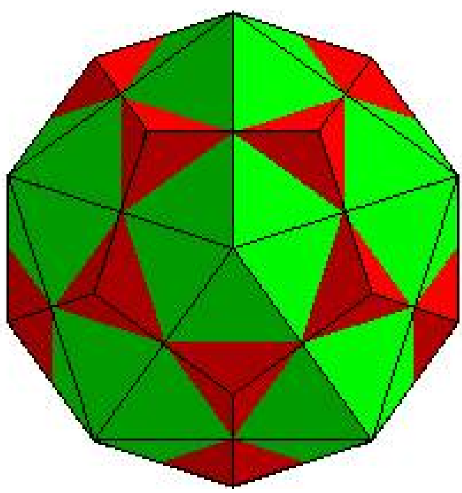

Figure 4 Compound of the regular dodecahedron and its dual, the regular icosahedron.

We would like to achieve something similar for a compound of Kirkman's Icosahedron and one of its duals. Therefore, we need to start by introducing the concept of the *polar set* of a set (see Lay [**11**]). If the inner product of $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ is given, as usual, by $x \cdot y = x_1y_1 + x_2y_2 + x_3y_3$, then the polar set K^* of any nonempty subset K of $\mathbb{R}^3$ is defined by

$$K^* = \left\{y \mid \text{for all } x \in K,\ x \cdot y \leq r^2\right\}.$$

Most authors require that $r = 1$ in the previous expression, but for our purposes it will be more convenient to use $r = 12$.

By means of two examples, let us show that polarity is a correspondence which relates each point to a plane and each plane to a point.

- When K consists of a single point, say $K = (x_1, x_2, x_3) = (9, 6, 6)$, then K^* is the closed half-space

$$9y_1 + 6y_2 + 6y_3 \leq 144,$$

which is bounded by the plane

$$9y_1 + 6y_2 + 6y_3 = 144.$$

- When K is a plane, say the one containing the pentagon with vertices $(9, 6, \pm 6)$, $(12, 4, 0)$, $(0, 12, \pm 8)$, then $K = \{(x_1, x_2, x_3) \mid 8x_1 + 12x_2 = 144\}$. In order to determine K^* we must find all those y for which

$$x_1 y_1 + x_2 y_2 + x_3 y_3 \leq 144 \quad \text{for all} \quad x \in K.$$

  It is easy to verify that in this case K^* is simply a ray (a half-line)

$$(y_1, y_2, y_3) = (8, 12, 0) + \lambda(-8, -12, 0) \quad \text{where } \lambda \geq 0$$

  having $(8, 12, 0)$ as its endpoint.

It should be clear that, in a completely analogous fashion, we can determine the polar sets of all the vertices and planes of Kirkman's Icosahedron (see TABLE 2 and TABLE 3).

TABLE 2: Vertices of P and their polar sets.

Coordinates of P	Face-planes of P^*
$(\pm 9, \pm 6, \pm 6)$	$\pm 9y_1 \pm 6y_2 \pm 6y_3 = 144$
$(\pm 12, \pm 4, 0)$	$\pm 12y_1 \pm 4y_2 = 144$
$(0, \pm 12, \pm 8)$	$\pm 12y_2 \pm 8y_3 = 144$
$(\pm 6, 0, \pm 12)$	$\pm 6y_1 \pm 12y_3 = 144$

TABLE 3: Faces of P and their polar sets.

Face-planes of P	Coordinates of P^*
$\pm 12x_1 \pm 6x_3 = 144$	$(\pm 12, 0, \pm 6)$
$\pm 8x_1 \pm 12x_2 = 144$	$(\pm 8, \pm 12, 0)$
$\pm 4x_2 \pm 12x_3 = 144$	$(0, \pm 4, \pm 12)$
$\pm 6x_1 \pm 6x_2 \pm 9x_3 = 144$	$(\pm 6, \pm 6, \pm 9)$

Our interest in the search for all the face-planes and coordinates of P^* is justified by the following basic result (see Lay [**11**]):

> If P is a bounded, convex polyhedron, which encloses the origin, then the polar set of P is itself a polyhedron P^* dual to P.

Before actually constructing the dual polyhedron P^*, we want to compare the coordinates for the vertices and also the equations for the face-planes of P and P^*. It is clear that

- whenever P has a vertex with coordinates (a, b, c), then P^* has a corresponding vertex with coordinates (c, b, a), and
- whenever P has a face-plane whose equation is $ax_1 + bx_2 + cx_3 = r^2$, then P^* has one given by $cy_1 + by_2 + ay_3 = r^2$.

This, in effect, means that by a rigid motion (reflection in the plane $x_1 = x_3$) we can bring the polyhedron P into coincidence with its dual P^*. Thus P and P^* are not just duals, but also isomorphic. They are in fact congruent. Hence duality and isomorphism occur simultaneously, a property that allows us to identify our polyhedron as being *self-dual* or *polar-syntypic with itself* or *autopolar* for short, as Kirkman would express it. For additional information on self-dual polyhedra, consult [**1, 4, 5, 10**]. In FIGURE 5 we have included three different images for the compound of Kirkman's Icosahedron and its dual.

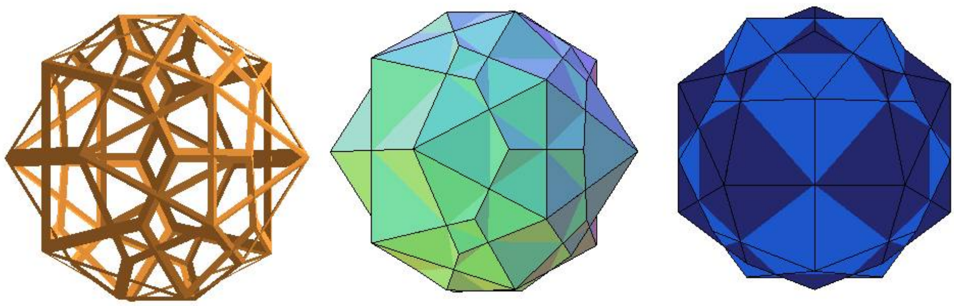

Figure 5 Compound of Kirkman's Icosahedron and its dual.

Closer inspection of the dodecahedron-icosahedron compound shown in FIGURE 4 reveals that the edges of both polyhedra intersect each other at right angles. It can be shown that they are, in fact, tangent to a so-called midsphere.

Does Kirkman's Icosahedron have a midsphere, and if so, what can we gain from it?

Fourth surprise

Let us consider two neighboring vertices of Kirkman's Icosahedron, say $P_0 = (9, 6, 6)$ and $P_1 = (6, 0, 12)$, and let M be the sphere with radius 12 centered at the origin. The line through the two points P_0 and P_1 is given by the parametric equation: $P(\lambda) = P_0 + \lambda(P_1 - P_0) = (9 - 3\lambda, 6 - 6\lambda, 6 + 6\lambda)$. Upon substitution into the equation for the sphere $x_1^2 + x_2^2 + x_3^2 = 144$, we get

$$9\lambda^2 - 6\lambda + 1 = (3\lambda - 1)^2 = 0.$$

Since this equation has two coincident real roots the line intersects the sphere in the unique point $(9 - 3\lambda, 6 - 6\lambda, 6 + 6\lambda)|_{\lambda=1/3} = (8, 4, 8)$, which means that the line is tangent to the sphere. Repeating this process for every pair of neighboring vertices, we obtain the set of contact points:

$$\left(\pm\frac{72}{7}, \pm\frac{36}{7}, \pm\frac{24}{7}\right) \qquad \left(\pm\frac{24}{7}, \pm\frac{36}{7}, \pm\frac{72}{7}\right)$$

$$\left(\pm\frac{72}{11}, \pm\frac{84}{11}, \pm\frac{72}{11}\right) \qquad (\pm 8, \pm 4, \pm 8)$$

$$(\pm 12, 0, 0) \qquad (0, \pm 12, 0) \qquad (0, 0, \pm 12)$$

So we conclude that all the edges of Kirkman's Icosahedron are tangent to the midsphere (see FIGURE 6). There is a "canonical" representation of this form for every

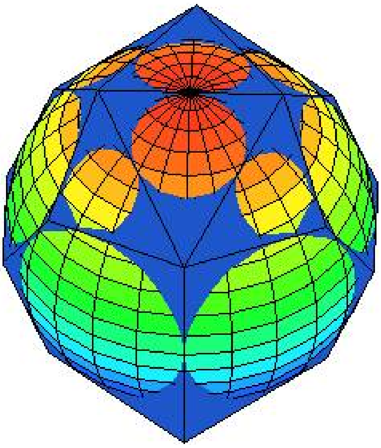

Figure 6 Kirkman's Icosahedron with midsphere.

polyhedron (Ziegler [**18**]). Moreover, in this case, the edges of the dual polyhedron are tangent to the sphere and, in fact, share the same edge-tangency (contact) points. In addition, the edges that correspond to each other under duality, intersect perpendicularly (for more details check Grünbaum [**6**], Sechelmann [**14**], and Ziegler [**17**]).

So, what are some of the consequences of our polyhedron having a midsphere M?

On the one hand, we immediately get two collections of non-overlapping circles on M's surface:

- the facet circles from the intersection of M with each of the faces of the polyhedron (referred to as a primal circle packing)
- the vertex horizon circles for each vertex (referred to as a dual circle packing).

A vertex horizon circle is the boundary of a spherical cap consisting of all those points on the sphere's surface which are visible from the respective vertex (see Sechelmann [**14**] and Ziegler [**18**]).

The circles from the primal circle packing touch if the corresponding faces are adjacent, and those from the dual circle packing touch if the corresponding vertices are adjacent (see FIGURE 7). Note that these two circle packings share the same edge-tangency points and that, moreover, they intersect orthogonally. This can be appreciated in FIGURE 7, which was obtained using the Koebe polyhedron editor developed by Sechelmann [**15**].

All those facet circles or incircles play an important role in the assignment of weights to the vertices that we used earlier in our treatment of the traveling salesman problem. Since two tangents drawn to a circle from an external point are always equal in length, the line segments from a vertex to the points where the incircle is tangent to the sides are congruent. Since the incircles touch if the corresponding faces are adjacent, we conclude that all the line segments from a vertex to each of its adjacent edge-tangency points are congruent. So this justifies assigning the weight $w(u) = |u - z|$ to the vertex u where z is any edge-tangency point on some incident edge.

Let us derive an additional consequence of having an assemblage of two interpenetrating polyhedra arranged around a midsphere. From the compound we are able to construct a pair of dual convex polyhedra, namely by considering

- the largest convex solid that is contained in it, and
- the smallest convex solid that contains it.

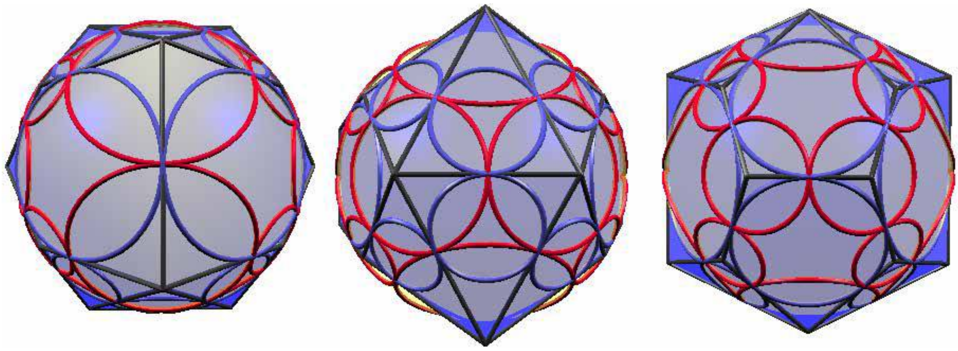

Figure 7 Kirkman's Icosahedron with facet (blue) and vertex horizon (red) circles.

Coxeter [**3**] refers to the first of these, which has the same face-planes as the compound as the core, and to the second, which has the same vertices as the compound as the case. The core and the case for the dodecahedron-icosahedron compound are well-known: They consist of the icosidodecahedron and the rhombic triacontahedron, respectively.

What can we say about the core and the case for the compound of Kirkman's Icosahedron and its dual? See FIGURE 8. First we remark that the core has all its vertices on the midsphere M, making it an inscribable polyhedron, whereas the case has all its faces tangent to the sphere M, making it a circumscribable polyhedron.

Second, since the vertices for the core are precisely the contact (edge-tangency) points and those for the case coincide with those of the compound, then it is fairly easy to see that we can get representations for both polyhedra having integral vertex coordinates!

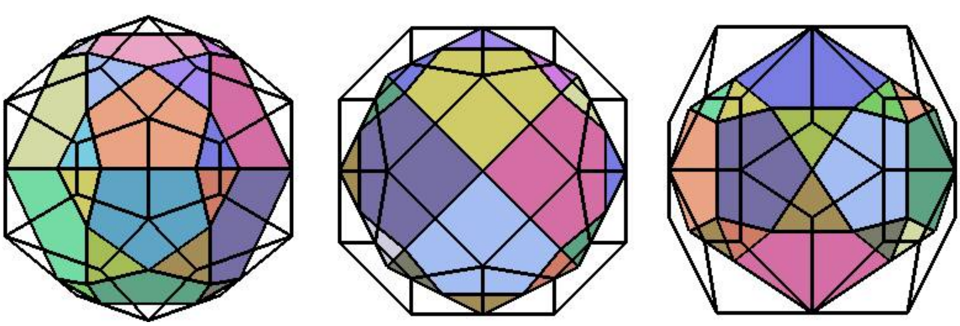

Figure 8 Case (transparent) and core (colored) for the compound of Kirkman's Icosahedron and its dual.

Finally, we want to consider a completely different question: Is this at all related with current research interests?

Fifth surprise

Surprisingly, we could say that the answer is yes. Richter [**12**], Rote [**13**], and Ziegler [**17**], for example, have been studying the problem of geometric realization for polyhedra which satisfy certain properties.

Interest has focused on the construction of convex polyhedra where either

(a) all of the vertex coordinates are small integers, or
(b) all of the edges are tangent to a sphere, or
(c) all of the edge lengths are integers.

In general, it is quite difficult to get such a polyhedral representation (see [**12, 13**]). On the other hand, Kirkman's Icosahedron satisfies all of these properties and some additional ones!

Considering that it all started with a simple straight-line drawing, the Schlegel diagram for the regular dodecahedron, a valid question is: How remarkable is Kirkman's Icosahedron?

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Summary The problem of geometric realization for convex polyhedra, which satisfy certain desirable properties, has received quite a bit of attention lately. Interest, mainly, has been on polyhedral representations where either all of the vertex coordinates are small integers, or all of the edge lengths are integers, or all of the edges are tangent to a sphere. In general, it is not easy to construct a convex polyhedron satisfying any of those criteria. We introduce a remarkable polyhedron that satisfies all of them.

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From Polygons to String Theory

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In this article we introduce a special kind of polygon called a reflexive polygon, and higher-dimensional generalizations called reflexive polytopes. In two dimensions, reflexive polygons are also called Fano polygons, after Gino Fano, an Italian mathematician born in 1871 who studied the relationship between geometry and modern algebra.

Two theoretical physicists, Maximilian Kreuzer and Harald Skarke, worked out a detailed description of three- and four-dimensional reflexive polytopes in the late 1990s. Why were physicists studying these polytopes? Their motivation came from string theory. Physicists use these polytopes to construct Calabi-Yau manifolds, which are geometric spaces that can model extra dimensions of our universe. A simple relationship between "mirror pairs" of polytopes corresponds to an extremely subtle connection between pairs of these geometric spaces. The quest to understand this connection has created the thriving field of mathematical research known as mirror symmetry.

What are reflexive polytopes? What does it mean to classify them? Why do they come in pairs? How can we build a complicated geometric space from a simple object like a triangle or a cube? And what does any of this have to do with physics? By answering these questions, we will uncover intricate relationships between combinatorics, geometry, and modern physics.

Classifying reflexive polygons

The points in the plane $\mathbb{R}^2$ with integer coordinates form a lattice, which we'll name N. A *lattice polygon* is a polygon whose vertices are in the lattice; in other words, lattice polygons have vertices with integer coordinates. We consider only convex polygons. An example of a convex lattice polygon is in FIGURE 1.

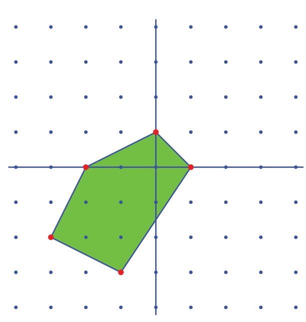

Figure 1 A lattice polygon

We say a lattice polygon is a *Fano polygon* if it has only one lattice point, the origin, in its interior.

How many Fano polygons are there? Can we list them all? The first step is to pull out our graph paper and try to draw a Fano polygon. A little experimentation will produce several Fano polygons, including triangles, quadrilaterals, and hexagons. Some examples are shown in FIGURE 2.

There are also ways to make new Fano polygons, once we find our first Fano polygon. For instance, we may rotate by 90 degrees or reflect across the x-axis. A more

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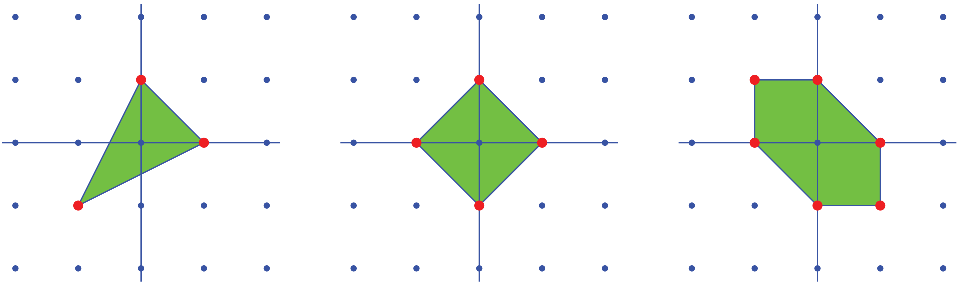


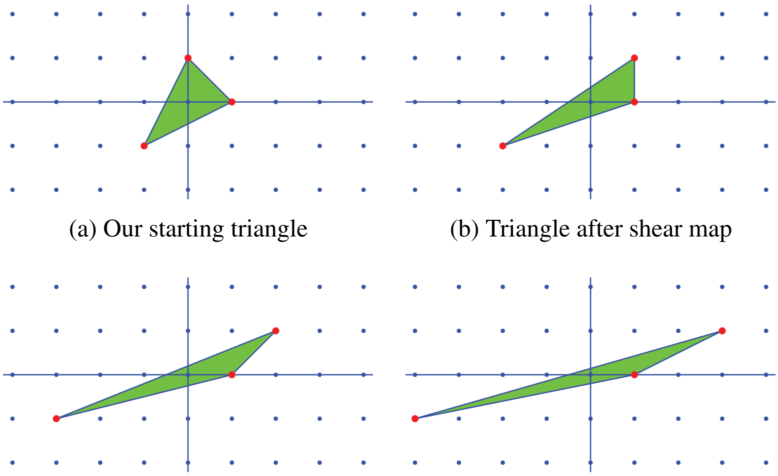
(a) (b) (c)

Figure 2 Fano triangle, quadrilateral, and hexagon

complicated type of map is the shear, which stretches a polygon in one direction. We can describe the shear using matrix multiplication: We map the point $\binom{x}{y}$ to

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix}.$$

In FIGURE 3 we illustrate the effects of this shear on a Fano triangle. Notice that after the shear, in FIGURE 3(b), there is still only one point in the interior of our triangle. Repeating the shear map, as seen in FIGURES 3(c) and 3(d), makes our triangle longer and skinnier. Iterating the shear map produces an infinite family of Fano triangles, each one longer and skinnier than the one before it.



(a) Our starting triangle (b) Triangle after shear map

(c) Triangle after two shears (d) Triangle after three shears

Figure 3 Effects of shears

This means that listing all Fano polygons is an impossible task! But we would still like to classify Fano polygons in some way. To do so, we shift our focus: Instead of counting individual Fano polygons, we will count types or classes of Fano polygons. We want two Fano polygons to belong to the same equivalence class if we can get from one to the other using reflections, rotations, and shears. Each map that we use should send lattice polygons to other lattice polygons.

Reflections, rotations, shears, and compositions of reflections, rotations, and shears are all linear transformations of the plane. In other words, they can be described by two-by-two matrices. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a matrix, where a, b, c, and d are real numbers. We let m_A be the "multiplication by A" map:

$$m_A \ : \ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

What conditions do we need to place on the matrix A to ensure that the corresponding map m_A sends Fano polygons to other Fano polygons? First, the map must send polygons to polygons, rather than squashing them into line segments. This means that A must be invertible, so that m_A is a one-to-one map from $\mathbb{R}^2$ to $\mathbb{R}^2$. Furthermore, m_A must send points with integer coordinates to other points with integer coordinates, so that lattice polygons go to lattice polygons. This will happen when A has integer entries.

The final condition on A is more subtle. We want two Fano polygons to belong to the same equivalence class if we can get from one to the other using rotations, reflections, and shears that map lattice polygons to lattice polygons. But our notion of equivalence should treat all Fano polygons equally: If two polygons $\diamond$ and $\diamond'$ are equivalent, it shouldn't matter whether we started with $\diamond$ and rotated or sheared it to form $\diamond'$, or started with $\diamond'$ and rotated or sheared it back to $\diamond$. In other words, our notion of equivalence must be symmetric. Now, if m_A sends $\diamond$ to $\diamond'$, the map that sends $\diamond'$ back to $\diamond$ is the map $m_{A^{-1}}$ defined by the inverse matrix A^{-1}. To ensure that $m_{A^{-1}}$ sends lattice polygons to lattice polygons, we require A^{-1} to have integer entries.

We can characterize A and A^{-1} using determinants. Because the product $A\,A^{-1}$ is the identity matrix, $(\det A)(\det A^{-1}) = 1$. But the determinant of a matrix with integer entries is an integer, so $\det A$ and $\det A^{-1}$ are both integers. The only way two integers can multiply to give 1 is for them to be both 1 or both -1, so $\det A = \pm 1$.

Matrices that satisfy this determinant property have a special name.

DEFINITION. $\mathbf{GL}(2, \mathbb{Z})$ is the set of two-by-two matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, which have integer entries and determinant $ad - bc$ equal to either 1 or -1.

The matrices in $\mathbf{GL}(2, \mathbb{Z})$ form a group, which is generated by rotation matrices, reflection matrices, and the shear matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Thus, the matrices in $\mathbf{GL}(2, \mathbb{Z})$ describe maps that are rotations, reflections, shears, or compositions of rotations, reflections, or shears.

By construction, for any matrix A in $\mathbf{GL}(2, \mathbb{Z})$, the map m_A is a continuous, one-to-one, and onto map of the plane, which restricts to a one-to-one and onto map from the lattice N to itself. The map sends lattice polygons to lattice polygons. In particular, the origin is the only lattice point in the interior of a Fano polygon, so it is the only lattice point that can be mapped to the interior of the image of a Fano polygon. Thus, this map sends Fano polygons to Fano polygons.

DEFINITION. We say two Fano polygons Δ and Δ' are $\mathbf{GL}(2, \mathbb{Z})$*-equivalent* (or sometimes just *equivalent*) if there exists a matrix A in $\mathbf{GL}(2, \mathbb{Z})$ such that $m_A(\Delta) = \Delta'$.

FIGURE 4 shows an example of equivalent polygons.

Now that we have a concept of equivalent Fano polygons, we can try again to describe the possible Fano polygons. How many $\mathbf{GL}(2, \mathbb{Z})$ equivalence classes of Fano polygons are there?

It turns out that there are only 16 equivalence classes of Fano polygons! A representative from each Fano polygon equivalence class is shown in FIGURE 5. (The figure seems to show twenty classes, but four of them are duplicated, for reasons that we will

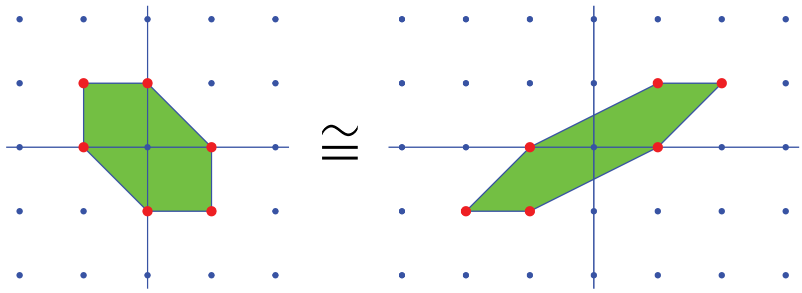


Figure 4 Two equivalent hexagons

investigate in the next section.) We will use the classification of Fano polygons later, but here we omit the proof; a combinatorial proof may be found in Nill [**7**].

Polar polygons The vertical arrows in FIGURE 5 indicate relationships between pairs of Fano polygons. In this section, we explain the correspondence. We start by asking a simple question: How can we describe a Fano polygon mathematically?

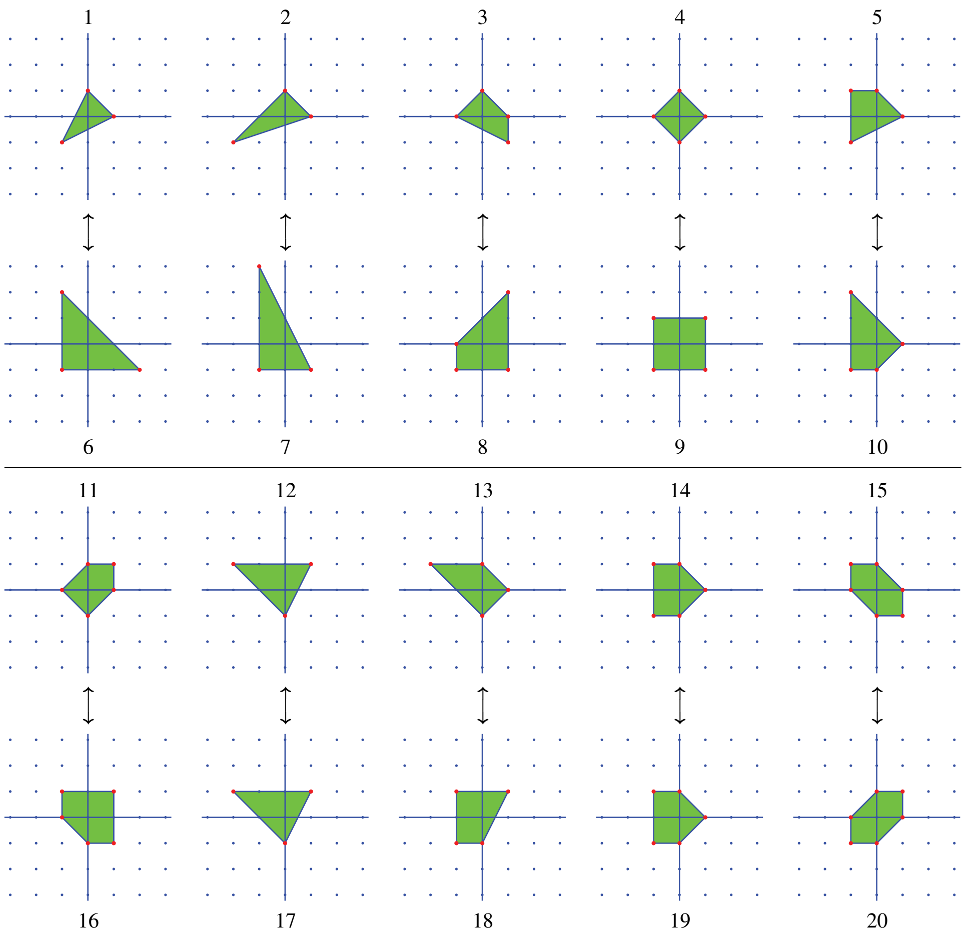


Figure 5 Classification of Fano polygons

One way to describe a polygon is to list the vertices. For instance, for the triangle in FIGURE 2(a), the vertices are (0, 1), (1, 0), and (−1, −1).

Each edge of a polygon is part of a line, so we can also describe a polygon by listing the equations of these lines. For the triangle in FIGURE 2(a), the equations are:

$$-x - y = -1,$$
$$2x - y = -1,$$
$$-x + 2y = -1.$$

Of course, there are many equivalent ways to write the equation for a line. We have chosen $ax + by = -1$ as our standard form. Any line that does not pass through the origin can be written in this way. Our standard form has the advantage that the whole triangle is described as the set of points (x, y) such that

$$-x - y \geq -1,$$
$$2x - y \geq -1,$$
$$-x + 2y \geq -1.$$

Notice that in the case of our Fano triangle, the coefficients a and b in our standard form for the equation of a line are all integers.

We want to use our edge equations to define a new polygon. We'd like our new polygon to live in its own copy of the plane. Let's call the set of points in this new plane that have integer coordinates M and name the new plane $M_{\mathbb{R}}$. Using the dot product, we can combine a point in our old plane, $N_{\mathbb{R}}$, with a point in our new plane, $M_{\mathbb{R}}$, to produce a real number

$$(n_1, n_2) \cdot (m_1, m_2) = n_1 m_1 + n_2 m_2.$$

If the point (n_1, n_2) lies in N and the point (m_1, m_2) lies in M, their dot product $(n_1, n_2) \cdot (m_1, m_2)$ is an integer.

Note that every dot product in this paper combines a vector in $N_{\mathbb{R}}$ (on the left) with a vector in $M_{\mathbb{R}}$ (on the right). The lattices N and M are isomorphic, as are the planes $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$, so the distinction between the two planes may seem artificial. However, as we will see later, the points in the N and M lattices play different roles in the physicists' construction.

Let's rewrite the edge equations of our Fano triangle using dot product notation:

$$(x, y) \cdot (-1, -1) = -1,$$
$$(x, y) \cdot (2, -1) = -1,$$
$$(x, y) \cdot (-1, 2) = -1.$$

The points (−1, −1), (2, −1), and (−1, 2) are the vertices of a new triangle in $M_{\mathbb{R}}$. We say that the new triangle, shown in FIGURE 6, is the *polar polygon* of our original triangle.

Suppose (n_1, n_2) is any point in our original triangle, and (m_1, m_2) is any point in its polar polygon. (These points do not need to be lattice points or boundary points; they might lie in the interiors of the triangles.) If we move the points around, the dot product $(n_1, n_2) \cdot (m_1, m_2)$ will vary continuously. The minimum possible dot product is −1, as we saw from our edge equations. Thus, the dot product of the two points will

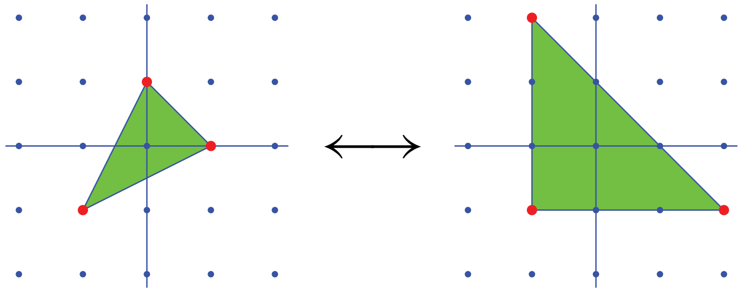


Figure 6 Our triangle and its polar polygon

satisfy the inequality

$$(n_1, n_2) \cdot (m_1, m_2) \geq -1.$$

We can use this inequality to define a polar polygon in general. Let Δ be any lattice polygon in our original plane $N_{\mathbb{R}}$ that contains $(0, 0)$. Formally, we say that the *polar polygon* Δ° is the polygon in $M_{\mathbb{R}}$ consisting of the points (m_1, m_2) such that

$$(n_1, n_2) \cdot (m_1, m_2) \geq -1$$

for all points (n_1, n_2) in Δ. We can find the vertices of Δ° using our standard-form equations for the edges of Δ, just as we did above.

The polar polygon of an arbitrary lattice polygon may fail to be a lattice polygon: The vertices might be rational numbers, rather than integers. On the other hand, the polar polygon of our example triangle is actually a lattice polygon; even better, it is itself a Fano polygon.

DEFINITION. Let Δ be a lattice polygon. If Δ° is also a lattice polygon, we say that Δ is *reflexive*.

If Δ is a reflexive polygon, then its polar polygon Δ° is also a reflexive polygon. We can repeat the polar polygon construction to find the polar polygon of Δ°. (For practice, try finding the polar polygon of the second triangle in FIGURE 6.) It turns out that the result is our original polygon:

$$(\Delta^\circ)^\circ = \Delta.$$

It's easy to see that every point in Δ is contained in $(\Delta^\circ)^\circ$, using the definition of a polar polygon. Showing that Δ and $(\Delta^\circ)^\circ$ are always identical is trickier; the proof (which we omit) depends on the fact that Δ is convex.

We say that a polygon Δ and its polar polygon Δ° are a *mirror pair*. The term *reflexive* also refers to this property: In a metaphorical sense, Δ and its polar polygon Δ° are reflections of each other. The fact that the Fano triangle in our example turned out to be reflexive is not a coincidence.

THEOREM 1. *A lattice polygon is reflexive if and only if it is Fano.*

Proof. First we show that every reflexive polygon is Fano. Let Δ be a reflexive polygon, and let (x, y) be a lattice point that lies strictly in the interior of Δ. Let $(v_1, w_1), (v_2, w_2), \ldots, (v_k, w_k)$ be the vertices of the polar polygon Δ°; because Δ is reflexive, the coordinates of each vertex are integers. We know that

$$(x, y) \cdot (v_i, w_i) \geq -1$$

for each vertex (v_i, w_i). Because (x, y) lies in the interior of Δ, the inequality must be strict:

$$(x, y) \cdot (v_i, w_i) > -1.$$

But (x, y) and (v_i, w_i) are both lattice points, so $(x, y) \cdot (v_i, w_i)$ must be an integer. Thus,

$$(x, y) \cdot (v_i, w_i) \geq 0.$$

Now, let a be any positive integer. The point $a(x, y) = (ax, ay)$ is a lattice point, and

$$\begin{aligned} a(x, y) \cdot (v_i, w_i) &\geq a \cdot 0 \\ &\geq -1. \end{aligned}$$

Because (ax, ay) satisfies the inequalities corresponding to each vertex of Δ°, we conclude that

$$(ax, ay) \cdot (m_1, m_2) \geq -1$$

for any point (m_1, m_2) in Δ°. Therefore, (ax, ay) is a point in $(\Delta^\circ)^\circ = \Delta$. If (x, y) is not $(0, 0)$, then we get an infinite number of different lattice points in Δ, one for each positive integer a. But this is impossible: Polygons are bounded, so they cannot contain infinite numbers of lattice points! Thus, (x, y) must be $(0, 0)$, so Δ is Fano.

We will use the classification of Fano polygons illustrated in FIGURE 5 to show that every Fano polygon is reflexive. The first step is to make sure that the representatives of Fano equivalence classes shown in the figure are reflexive polygons. The vertical arrows in FIGURE 5 connect each of the illustrated polygons to its polar dual (checking this fact is a fun exercise!). The computation tells us that every Fano polygon is equivalent to a reflexive polygon. Notice that the Fano polygons numbered 12 through 15 in FIGURE 5 are self-dual: Their polar duals are equivalent to the original polygons. (Can you find the matrix that sends polygon 13 to its polar dual?)

To finish the proof, we must show that if Γ is equivalent to a reflexive polygon Δ, then Γ is also a reflexive polygon. Let Δ be a reflexive polygon, let A be a matrix in $\mathbf{GL}(2, \mathbb{Z})$, and suppose that $\Gamma = m_A(\Delta)$. Let $B = (A^{\mathrm{T}})^{-1}$, the inverse of the transpose matrix of A. Taking the transpose of a matrix does not change its determinant, and the inverse of a matrix in $\mathbf{GL}(2, \mathbb{Z})$ is also in $\mathbf{GL}(2, \mathbb{Z})$, so B is another member of $\mathbf{GL}(2, \mathbb{Z})$. We claim that $\Gamma^\circ = m_B(\Delta^\circ)$.

Let (n_1, n_2) be a point in Γ, and let (m_1, m_2) be any point in $M_{\mathbb{R}}$. We know that

$$\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = A \begin{pmatrix} n_1' \\ n_2' \end{pmatrix}$$

for some point (n_1', n_2') in Δ. Let

$$\begin{pmatrix} m_1' \\ m_2' \end{pmatrix} = B^{-1} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix},$$

so that

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = B \begin{pmatrix} m_1' \\ m_2' \end{pmatrix}.$$

Now,

$$
\begin{aligned}
(n_1, n_2) \cdot (m_1, m_2) &= \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \\
&= \left(A \begin{pmatrix} n_1' \\ n_2' \end{pmatrix} \right)^{\mathrm{T}} B \begin{pmatrix} m_1' \\ m_2' \end{pmatrix} \\
&= \begin{pmatrix} n_1' \\ n_2' \end{pmatrix}^{\mathrm{T}} A^{\mathrm{T}} B \begin{pmatrix} m_1' \\ m_2' \end{pmatrix} \\
&= \begin{pmatrix} n_1' \\ n_2' \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} m_1' \\ m_2' \end{pmatrix} \\
&= (n_1', n_2') \cdot (m_1', m_2').
\end{aligned}
$$

We see that $(n_1, n_2) \cdot (m_1, m_2) \geq -1$ if and only if $(n_1', n_2') \cdot (m_1', m_2') \geq -1$, and therefore (m_1, m_2) is in Γ° if and only if (m_1', m_2') is in Δ°. Thus, $\Gamma^\circ = m_B(\Delta^\circ)$. Since Δ° is a lattice polygon, $m_B(\Delta^\circ)$ must be a lattice polygon, so Γ is reflexive. ■

A reflexive polygon and its polar dual are intricately related. It's pretty easy to see that a polygon and its polar dual have the same number of sides and vertices. Other connections are more subtle. For instance, the number of lattice points on the boundary of a reflexive polygon and the number of lattice points on the boundary of its polar dual always add up to twelve! For the polygons in FIGURE 6, the computation is $3 + 9 = 12$. We can check that this holds in general by counting points in FIGURE 5 and using the fact that equivalent Fano polygons have the same number of lattice points. (See [**8**] for other proofs that the boundary points add to twelve, which use combinatorics, algebraic geometry, and number theory.)

Higher dimensions

Let's extend the idea of Fano and reflexive polygons to dimensions other than 2. In order to do so, we need to describe the k-dimensional generalizations of polygons, which we will call *polytopes*. There are several ways to do this. We take the point of view that polygons are described by writing down a list of vertices, adding line segments that connect these vertices, and then filling in the interior of the polygon. Similarly, in k dimensions, our intuition suggests that we should describe a polytope by writing down a list of vertices, connecting them, and then filling in the inside. The formal definition is as follows.

DEFINITION. Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_q\}$ be a set of points in $\mathbb{R}^k$. The *polytope* with vertices $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_q\}$ is the set of points of the form

$$\vec{x} = \sum_{i=1}^{q} t_i \vec{v}_i,$$

where the t_i are nonnegative real numbers satisfying $t_1 + t_2 + \cdots + t_q = 1$. (The polytope is called the *convex hull* of the points $\vec{v}_i$.)

We illustrate a three-dimensional polytope in FIGURE 7.

Let N be the lattice of points with integer coordinates in $\mathbb{R}^k$; we refer to this copy of $\mathbb{R}^k$ as $N_{\mathbb{R}}$. A *lattice polytope* is a polytope whose vertices lie in N.

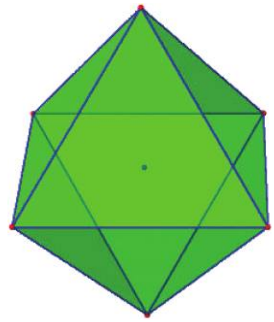


Figure 7 A three-dimensional polytope

The one-dimensional case To understand the definition of a lattice polytope better, let's consider the one-dimensional polytope that has vertices 1 and -1. Formally, this polytope consists of all points on the real number line that can be written as $1 \cdot t_1 + -1 \cdot t_2$, where $t_1 \geq 0$, $t_2 \geq 0$, and $t_1 + t_2 = 1$. We can visualize this by imagining an ant walking on the number line. Our ant starts at 0; for a fraction of an hour the ant walks to the right, toward the point 1, then for the rest of the hour the ant walks left, toward the point -1. The ant can reach any point in the closed interval $[-1, 1]$, so all of these points belong to our lattice polytope. This one-dimensional lattice polytope is shown in FIGURE 8.

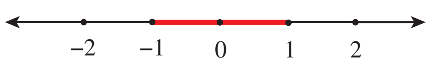


Figure 8 A one-dimensional polytope

The *k*-dimensional case Just as we did in two dimensions, we can define a *dual lattice* M in k dimensions by taking a new copy of $\mathbb{R}^k$, which we'll refer to as $M_{\mathbb{R}}$, and letting M be the points with integer coordinates in $M_{\mathbb{R}}$. The dot product pairs points in $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ to produce real numbers:

$$(n_1, \ldots, n_k) \cdot (m_1, \ldots, m_k) = n_1 m_1 + \cdots + n_k m_k.$$

If we take the dot product of a point in our original lattice N and a point in our dual lattice M, we obtain an integer.

We can use our k-dimensional dot product to define polar polytopes. If Δ is a lattice polytope in N that contains the origin, we say its *polar polytope* Δ° is the polytope in M given by

$$\{(m_1, \ldots, m_k) : (n_1, \ldots, n_k) \cdot (m_1, \ldots, m_k) \geq -1 \text{ for all } (n_1, \ldots, n_k) \in \Delta\}.$$

We say that a lattice polytope is *Fano* if the only lattice point that lies strictly in its interior is the origin, and that a lattice polytope containing the origin is *reflexive* if its polar polytope is also a lattice polytope. Just as in two dimensions, we find that the polar of the polar of a reflexive polytope is the original polytope ($(\Delta^\circ)^\circ = \Delta$), and we say that a reflexive polytope Δ and its polar polytope Δ° are a *mirror pair*. We illustrate a mirror pair of three-dimensional polytopes in FIGURE 9.

Every reflexive polytope is a Fano polytope: The proof that we used to show reflexive polygons are Fano carries through in k dimensions.

However, not every Fano polytope is reflexive. We can construct an example starting with the cube in FIGURE 9. The cube with vertices at $(\pm 1, \pm 1, \pm 1)$ is both Fano and

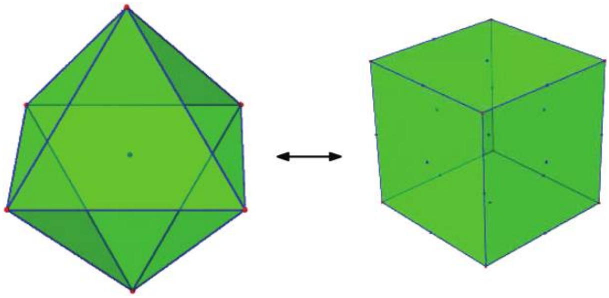


Figure 9 The octahedron and cube: a mirror pair

reflexive. But we can form a new polytope from the cube by removing the vertex at $(1, 1, 1)$ and taking the convex hull of the remaining lattice points. The resulting polytope is shown in FIGURE 10. It is clearly Fano, since the origin lies in the interior, and we have not added any lattice points. However, it is not reflexive. The equation of the face spanned by the new vertices $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$ is $x + y + z = 2$, or $-\frac{1}{2}x + -\frac{1}{2}y + -\frac{1}{2}z = -1$ in standard form. Thus, the polar polygon has $(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ as a vertex, so it is not a lattice polytope.

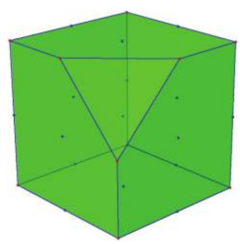


Figure 10 A Fano polytope, which is not reflexive

From now on, we will focus on the special properties of reflexive polytopes.

How many equivalence classes of reflexive polytopes are there in dimension n? In one dimension there is only one reflexive polytope, namely, the closed interval $[-1, 1]$. (It follows that the one-dimensional reflexive polytope is its own polar polytope.) We have seen the 16 equivalence classes of two-dimensional reflexive polygons. The physicists, Maximilian Kreuzer and Harald Skarke, counted equivalence classes of reflexive polytopes in dimensions three and four. Their results are summarized in TABLE 1; a description of a representative polytope from each class may be found at [**6**].

TABLE 1: Counting Reflexive Polytopes

Dimension	**Classes of Reflexive Polytopes**
1	1
2	16
3	4,319
4	473,800,776
≥ 5	??

The physicists' method for classifying polytopes was very computationally intensive, so it is not effective in higher dimensions. In dimensions five and higher, the number of equivalence classes of reflexive polytopes is an open problem.

The connection to string theory

String theory and mirror families Why were physicists classifying reflexive polytopes? As we noted in the introduction, the answer lies in a surprising prediction made by string theory.

String theory is one candidate for what physicists call a *Grand Unified Theory*, or GUT for short. A Grand Unified Theory would unite the theory of general relativity with the theory of quantum physics. General relativity is an effective description for the way our universe works on a very large scale, at the level of stars, galaxies, and black holes. The theory of quantum physics, on the other hand, describes the way our universe works on a very small scale, at the level of electrons, quarks, and neutrinos. Attempts to combine the theories have failed: standard methods for "quantizing" physical theories don't work when applied to general relativity, because they predict that empty space should hold infinite energy.

String theory solves the infinite energy problem by re-defining what a fundamental particle should look like. We often imagine electrons as point particles, that is, zero-dimensional objects. According to string theory, we should treat the smallest components of our universe as one-dimensional objects called *strings*. Strings can be open, with two endpoints, or they can be closed loops. They can also vibrate with different amounts of energy. The different vibration frequencies produce all the particles that particle physicists observe: quarks, electrons, photons, and so forth.

We are accustomed to thinking of point particles as located somewhere in four dimensions of space and time. But string theory insists on something more. To be consistent, string theory requires that strings extend beyond the familiar four dimensions, into extra dimensions. The extra dimensions must have particular geometric shapes. Mathematically, these shapes are known as *Calabi-Yau manifolds*.

To construct a string-theoretic model of the universe, we must choose a particular Calabi-Yau manifold to represent the extra dimensions at a point in four-dimensional space-time. Further, there are multiple ways to use this manifold. We will consider two of these theories, the *A-model* and the *B-model*. These models have very similar definitions: The only difference is whether one works with a particular variable or its complex conjugate. However, the physical consequences predicted by the A-model and the B-model for a particular geometric space are quite different.

When physicists began to explore the implications of string theory, they stumbled on a surprising correspondence: Different sets of choices can yield the same observable physics. As physicists studied the A-model and the B-model for many Calabi-Yau manifolds, they discovered pairs of Calabi-Yau manifolds where the A-model for the first manifold in the pair made the same predictions as the B-model for the second manifold, and vice versa. They hypothesized that a *mirror manifold* should exist for every Calabi-Yau manifold (or at least every Calabi-Yau manifold where the A-model and B-model can be defined).

In more mathematical terms, the physicists' hypothesis implies that Calabi-Yau manifolds should arise in paired or *mirror* families. (Mathematicians prefer to work with multi-parameter families of Calabi-Yau manifolds, rather than individual manifolds, because moving from one point of four-dimensional space-time to a nearby point might deform the shape of the attached Calabi-Yau manifold.) Since the geometric properties of each family determine the same physical theories, we can use information about the geometry of one family to study the properties of the mirror spaces.

What does this have to do with reflexive polytopes?

We can use reflexive polytopes to describe mirror families! To do so, we need a recipe that starts with a reflexive polytope and that produces a geometric space. We will proceed in two steps: First, we will use our polytope to build a family of polynomials, and then we will use our polynomials to describe a family of geometric spaces. (Technically, we will work with Laurent polynomials, which can involve negative powers.) We will obtain a mirror pair of families of geometric spaces corresponding to each mirror pair of polytopes:

$$\begin{array}{ccc}
\text{polytope} & \longleftrightarrow & \text{polar polytope} \\
\downarrow & & \downarrow \\
\text{Laurent polynomial} & \longleftrightarrow & \text{mirror Laurent polynomial} \\
\downarrow & & \downarrow \\
\text{family of spaces} & \longleftrightarrow & \text{mirror family}
\end{array}$$

From polytopes to polynomials Let Δ be a reflexive polytope. We want to construct a family of polynomials using Δ. We start by defining the variables for our polynomial. We do so by associating the variable z_i to the ith standard basis vector in the lattice N:

$$\begin{aligned}
(1, 0, \dots, 0) &\leftrightarrow z_1 \\
(0, 1, \dots, 0) &\leftrightarrow z_2 \\
&\vdots \\
(0, 0, \dots, 1) &\leftrightarrow z_n
\end{aligned}$$

Think of the z_i as complex variables: We will let ourselves substitute any nonzero complex number for z_i.

Next, for each lattice point in the polar polytope Δ°, we define a monomial, using the following rule:

$$(m_1, \dots, m_k) \leftrightarrow z_1^{m_1} z_2^{m_2} \cdots z_k^{m_k}.$$

Finally, we multiply each monomial by a complex parameter α_j, and add up the monomials. This gives us a family of polynomials parameterized by the α_j.

Let's work out what this step looks like in the case of the one-dimensional reflexive polytope $\Delta = [-1, 1]$. Because we are working with a one-dimensional lattice N, there is only one standard basis vector, namely 1. Corresponding to this basis vector, we have one monomial, z_1. Next we consider the polar polytope Δ°. The one-dimensional reflexive polytope is its own polar dual, so $\Delta^\circ = [-1, 1]$. Thus, Δ° has three lattice points, -1, 0, and 1. From each of these lattice points, we build a monomial, as follows:

$$\begin{aligned}
-1 &\mapsto z_1^{-1} \\
0 &\mapsto 1 \\
1 &\mapsto z_1
\end{aligned}$$

Finally, we multiply each monomial by a complex parameter and add the results. We obtain the family of Laurent polynomials $\alpha_1 z_1^{-1} + \alpha_2 + \alpha_3 z_1$, which depends on the three parameters α_1, α_2, and α_3. Notice that, because z_1 is raised to a negative power in the first term, we cannot allow z_1 to be zero.

Next, let's look at the family of Laurent polynomials corresponding to the reflexive triangle in FIGURE 2(a). We are now working with a two-dimensional polytope, so we have two variables, z_1 and z_2. The polar polygon of our reflexive triangle is shown in FIGURE 6. It contains ten lattice points (including the origin), so we will have ten monomials.

$$\begin{aligned}(-1,2) &\mapsto z_1^{-1}z_2^2\\ (-1,1) &\mapsto z_1^{-1}z_2\\ (0,1) &\mapsto z_2\\ &\vdots\\ (2,-1) &\mapsto z_1^2 z_2^{-1}\end{aligned}$$

When we multiply each monomial by a complex parameter and add the results, we obtain a family of Laurent polynomials of the form

$$\alpha_1 z_1^{-1}z_2^2 + \alpha_2 z_1^{-1}z_2 + \alpha_3 z_2 + \alpha_4 z_1^{-1} + \alpha_5 + \alpha_6 z_1 \\ + \alpha_7 z_1^{-1}z_2^{-1} + \alpha_8 z_2^{-1} + \alpha_9 z_1 z_2^{-1} + \alpha_{10} z_1^2 z_2^{-1}.$$

The mirror family of polynomials is obtained from the big reflexive triangle in FIGURE 6. We are still working in two dimensions, so we still need two variables; let's call these w_1 and w_2. The big triangle's polar polygon is the triangle in FIGURE 2(a), since the polar of the polar dual of a polygon is the original polygon. Thus, the mirror family of polynomials will only have four terms, corresponding to the four lattice points of the triangle in FIGURE 2(a). It is given by

$$\beta_1 w_1^{-1}w_2^{-1} + \beta_2 w_2 + \beta_3 + \beta_4 w_1.$$

From polynomials to spaces If we set a Laurent polynomial equal to zero, the resulting solutions describe a geometric space. Let's look at some examples using the family $\alpha_1 z_1^{-1} + \alpha_2 + \alpha_3 z_1$ obtained from the one-dimensional reflexive polytope. If we set $\alpha_1 = -1$, $\alpha_2 = 0$, and $\alpha_3 = 1$, we obtain the polynomial $-z_1^{-1} + z_1 = 0$. Solving, we find that $z_1^2 = 1$, so the solutions are the pair of points 1 and -1. If we set $\alpha_1 = 1$, $\alpha_2 = 0$, and $\alpha_3 = 1$, we obtain the polynomial $z_1^{-1} + z_1 = 0$. In this case, we find that $z_1^2 = -1$, so the solutions are the pair of points i and $-i$. (Now we see why it is important to work over the complex numbers!)

As we vary the parameters α_1, α_2, and α_3, we will obtain all pairs of nonzero points in the complex plane. Since the one-dimensional reflexive polytope is its own polar dual, the mirror family will also correspond to pairs of nonzero points in the complex plane. These are zero-dimensional geometric spaces inside a one-complex-dimensional ambient space. To describe more interesting geometric spaces, we'll have to increase dimensions.

What are the spaces corresponding to the mirror pair of triangles in FIGURE 6? Let's set the Laurent polynomials corresponding to the smaller triangle equal to zero.

$$\alpha_1 z_1^{-1}z_2^2 + \alpha_2 z_1^{-1}z_2 + \alpha_3 z_2 + \alpha_4 z_1^{-1} + \alpha_5 + \alpha_6 z_1 \\ + \alpha_7 z_1^{-1}z_2^{-1} + \alpha_8 z_2^{-1} + \alpha_9 z_1 z_2^{-1} + \alpha_{10} z_1^2 z_2^{-1} = 0.$$

We can multiply through by z_1z_2 without changing the nonzero solutions. We obtain

$$\alpha_1 z_2^3 + \alpha_2 z_2^2 + \alpha_3 z_1 z_2^2 + \alpha_4 z_2 + \alpha_5 z_1 z_2 + \alpha_6 z_1^2 z_2 + \alpha_7 + \alpha_8 z_1 + \alpha_9 z_1^2 + \alpha_{10} z_1^3 = 0.$$

Let's re-order, so that terms of higher degree come first:

$$\alpha_{10} z_1^3 + \alpha_1 z_2^3 + \alpha_6 z_1^2 z_2 + \alpha_3 z_1 z_2^2 + \alpha_9 z_1^2 + \alpha_2 z_2^2 + \alpha_5 z_1 z_2 + \alpha_8 z_1 + \alpha_4 z_2 + \alpha_7 = 0.$$

As we vary our parameters α_i, we will obtain all possible degree-three or *cubic* polynomials in two complex variables. We cannot graph the solutions to these polynomials, because they naturally live in two complex (or four real) dimensions. However, we can graph the solutions that happen to be pairs of real numbers. These will trace out a curve in the plane. The real solutions for two possible choices of the parameters α_i are shown in FIGURES 11 and 12.

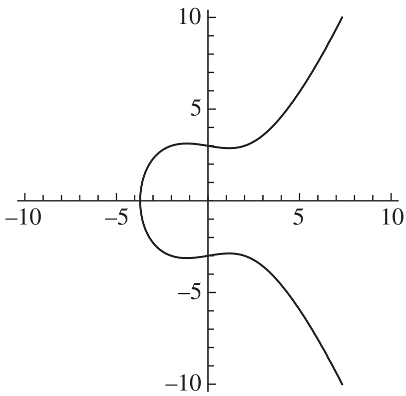


Figure 11 A real cubic curve

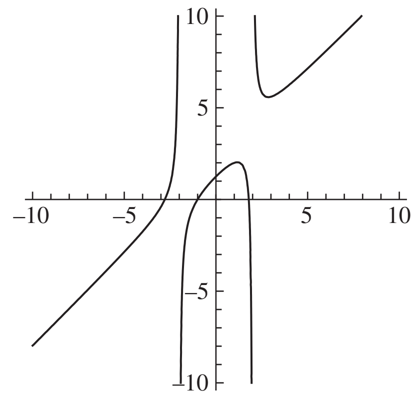


Figure 12 Another cubic curve

The mirror family of spaces is given by solutions to

$$\beta_1 w_1^{-1} w_2^{-1} + \beta_2 w_2 + \beta_3 + \beta_4 w_1 = 0.$$

We can multiply through by w_1w_2 without changing the nonzero solutions:

$$\beta_1 + \beta_2 w_1 w_2^2 + \beta_3 w_1 w_2 + \beta_4 w_1^2 w_2 = 0.$$

Our mirror family of spaces also consists of solutions to cubic polynomials, but instead of taking all possible cubic polynomials, we have a special subfamily.

Physicists are particularly interested in *Calabi-Yau threefolds*: These three complex-dimensional (or six real-dimensional) spaces are candidates for the extra dimensions of the universe. We can generate Calabi-Yau threefolds using reflexive polytopes. For instance, one of the spaces in the family corresponding to the polytope with vertices $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, and $(-1, -1, -1, -1)$ can be described by the polynomial

$$z_1^5 + z_2^5 + z_3^5 + z_4^5 + 1 = 0.$$

Although we cannot graph this six-dimensional space, we can begin to understand its complexity by drawing a two-dimensional slice in $\mathbb{R}^3$. One possible slice is shown in FIGURE 13. You can generate and rotate slices of this space at the website [**4**].

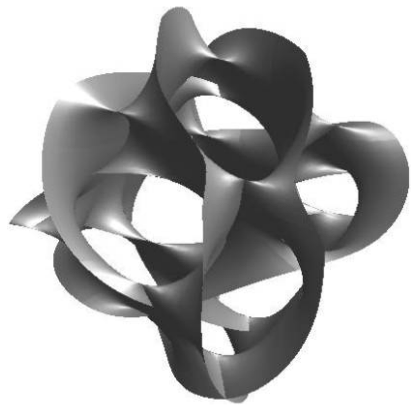


Figure 13 Slice of a Calabi-Yau threefold

Physical, combinatorial, and geometrical dualities String theory inspired physicists to study the geometric spaces known as Calabi-Yau manifolds. Using the duality between pairs of reflexive polytopes, we have written a recipe for constructing mirror families of these manifolds:

$$\begin{array}{ccc} \text{polytope} & \longleftrightarrow & \text{polar polytope} \\ \big\downarrow & & \big\downarrow \\ \text{Laurent polynomials} & \longleftrightarrow & \text{mirror Laurent polynomials} \\ \big\downarrow & & \big\downarrow \\ \text{spaces} & \longleftrightarrow & \text{mirror spaces} \end{array}$$

The ingredients in our recipe are combinatorial data, such as the number of points in a lattice polytope; the results of our recipe are paired geometric spaces. Combinatorics not only allows us to cook up these spaces, it gives us a way to study them: We can investigate geometric and topological properties of Calabi-Yau manifolds by measuring the properties of the polytopes we started with.

The geometric properties of a Calabi-Yau manifold V are encoded in a list of nonnegative integers known as *Hodge numbers*. (Readers who have studied algebraic topology will recognize the vector spaces $H^n(V, \mathbb{C})$, whose dimensions are given by the *Betti numbers*; the Hodge numbers tell us how to break up these vector spaces into smaller subspaces, using results from complex analysis.)

Two of the Hodge numbers, $a(V)$ and $b(V)$, count ways in which V can be deformed. In the physicists' language, the Hodge number $a(V)$ counts the number of *A-model variations*; mathematically, these are the number of independent ways to deform the notion of distance, or *Kähler metric*, on V. The Hodge number $b(V)$ counts the number of *B-model variations*, that is, the ways to deform the *complex structure* of V. (The complex structure tells us how to find local coordinate patches for V that look like subspaces of $\mathbb{C}^k$.)

Before physicists arrived on the scene, these two types of deformations were the provinces of two different fields of mathematics: Differential geometers studied distance and metrics, using the tools of differential equations, while algebraic geometers studied complex structures, relying on the power of modern algebra. Mirror symmetry predicts that, for Calabi-Yau manifolds, these deformations are intimately related. If V and V° are a mirror pair of Calabi-Yau manifolds, then their possible A-model variations and B-model variations must be reversed. Physicists conjectured that, given a Calabi-Yau manifold V, we should be able to find a mirror manifold V° with the appropriate Hodge numbers:

$$a(V) = b(V^\circ) \quad \text{and} \quad b(V) = a(V^\circ).$$

When physicists first framed this conjecture, very few examples of Calabi-Yau manifolds were known. Reflexive polytopes provide both a rich source of example Calabi-Yau manifolds, and a concrete mathematical construction of their mirrors. In the early 1990s, Victor Batyrev discovered and proved formulas for $a(V)$ and $b(V)$, which work when V is obtained from a reflexive polytope Δ of dimension $k \geq 4$:

$$a(V) = \ell(\Delta) - k - 1 - \sum_{\Gamma} \ell^*(\Gamma) + \sum_{\Theta} \ell^*(\Theta)\ell^*(\hat{\Theta})$$

$$b(V) = \ell(\Delta^\circ) - k - 1 - \sum_{\Gamma^\circ} \ell^*(\Gamma^\circ) + \sum_{\Theta^\circ} \ell^*(\Theta^\circ)\ell^*(\hat{\Theta}^\circ).$$

Here $\ell()$ is the number of lattice points in a polytope or face, and $\ell^*()$ is the number of lattice points in the interior of a polytope or face. (For a face, this means that $\ell^*()$ does not count lattice points on its boundary.) The Γ are codimension 1 faces of Δ (that is, faces of dimension $k-1$), Θ are codimension 2 faces of Θ (that is, faces of dimension $k-2$), and $\hat{\Theta}$ is the face of Δ° dual to Θ; similarly, Γ° are codimension 1 faces of Δ°, Θ° are codimension 2 faces of Θ°, and $\hat{\Theta}^\circ$ is the face of Δ dual to Θ°. Notice that the variations of complex structure are controlled by the number of lattice points in the polar polytope Δ°. This is reasonable, because each lattice point in Δ° corresponds to a monomial in the equation for V, and in turn the equation for V determines a complex structure.

Since $(\Delta^\circ)^\circ = \Delta$, it follows immediately that the Calabi-Yau manifolds V° obtained from Δ° satisfy

$$a(V) = b(V^\circ) \text{ and } b(V) = a(V^\circ).$$

Thus, Batyrev was able to use reflexive polytopes to turn a conjecture motivated by physics into a solid mathematical theorem.

In this article, we constructed Calabi-Yau manifolds as $(k-1)$-dimensional spaces described by a single equation in k complex variables. We can generalize this construction to build $(k-r)$-dimensional Calabi-Yau manifolds described by a system of r equations in k variables. The starting point is a reflexive polytope Δ. To define the r equations, we must divide the vertices of Δ into r disjoint subsets. The subsets of vertices define polytopes of lower dimension, and we can guarantee that the corresponding equations describe a Calabi-Yau manifold by placing conditions on these polytopes. Lev Borisov introduced a method for constructing a mirror Calabi-Yau manifold, corresponding to r more lower-dimensional polytopes. Together, Batyrev and Borisov proved a very general form of the equality

$$a(V) = b(V^\circ) \quad \text{and} \quad b(V) = a(V^\circ).$$

Their proof was indirect: They used the combinatorial data of the various polytopes to define a polynomial, and then showed that the coefficients of this polynomial correspond to Hodge numbers. (See [**1**] for an overview of this material.) Very recently, in

[**2**], the first author and a collaborator found closed-form expressions for a and b for Calabi-Yau threefolds described by two equations; these formulas directly generalize Batyrev's lattice point counting formulas from the single-equation case.

Reflexive polytopes remain the most bountiful source for examples of Calabi-Yau manifolds, and the simple combinatorial duality between polar polytopes continues to provide insight into geometrical dualities (mirror spaces) and even physical dualities (mirror universes)!

Acknowledgments The authors thank Andrey Novoseltsev for assistance in generating images of reflexive polytopes. We are grateful to Michael Orrison and the anonymous referees for their thoughtful and perceptive suggestions. Charles Doran thanks Bard College and the Claremont Colleges Colloquium for the opportunity to present this material. Ursula Whitcher gratefully acknowledges the partial support of her postdoctoral fellowship at Harvey Mudd College by the National Science Foundation under the Grant DMS-083996.

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Summary We describe special kinds of polygons, called Fano polygons or reflexive polygons, and their higher-dimensional generalizations, called reflexive polytopes. Pairs of reflexive polytopes are related by an operation called polar duality. This combinatorial relationship has a deep and surprising connection to string theory: One may use reflexive polytopes to construct "mirror" pairs of geometric spaces called Calabi-Yau manifolds that could represent extra dimensions of the universe. Reflexive polytopes remain a rich source of examples and conjectures in mirror symmetry.

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Proof Without Words: Sums of Cubes

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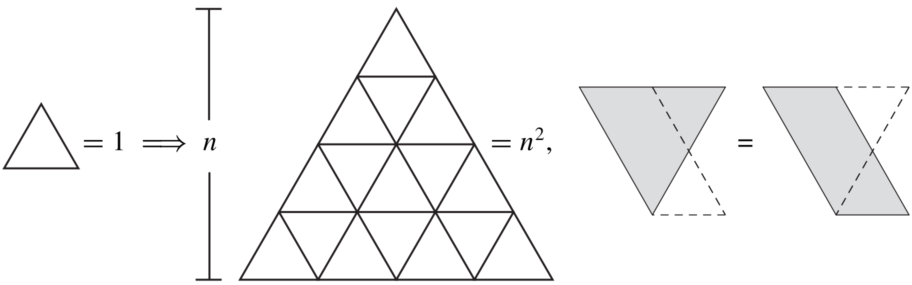

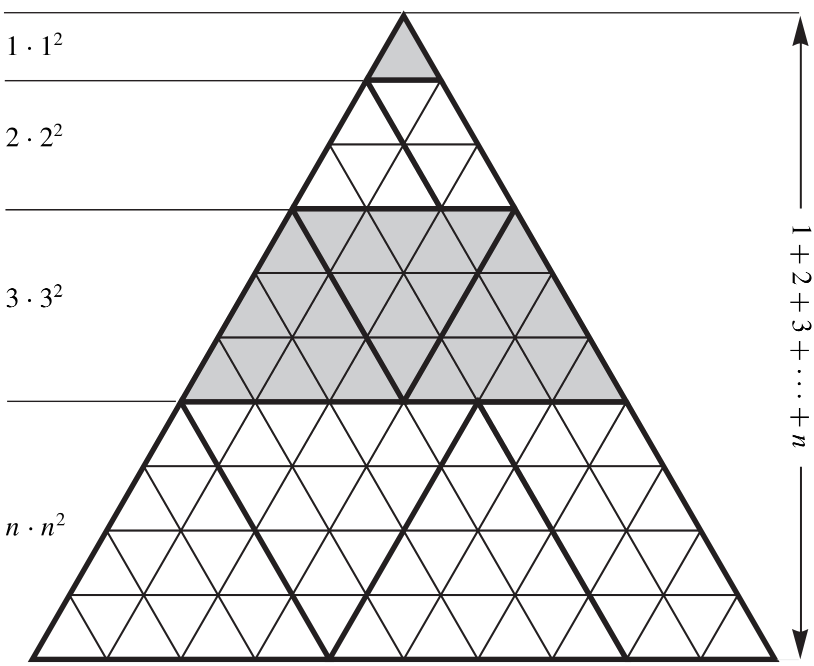

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$

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NOTES

A Short Elementary Proof of $\sum 1/k^2 = \pi^2/6$

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The aim of this note is to give a truly elementary proof of the identity

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, \tag{1}$$

which can be appreciated by anyone who understands elementary calculus. The identity (1) is often referred to as the "Basel Problem" and was solved by Euler around 1735. More on the interesting history can be found in [**5, 15**].

The idea in this paper is to derive an explicit formula for the partial sums of (1) by rewriting it as a telescoping sum. For that we exploit recursion relations between the integrals

$$A_n = \int_0^{\pi/2} \cos^{2n} x \, dx \qquad \text{and} \qquad B_n = \int_0^{\pi/2} x^2 \cos^{2n} x \, dx$$

for $n \geq 0$. In particular we derive the explicit estimate

$$0 \leq \frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2} = 2\frac{B_n}{A_n} \leq \frac{\pi^2}{4(n+1)} \tag{2}$$

from which (1) follows by letting $n \to \infty$. The idea is similar to the one by Masuoka [**13**], but the estimate of the remainder term is even simpler. An alternative way to write (1) as a telescoping sum is given in [**2**].

There are many short proofs of (1), but most rely on additional knowledge. A nice collection is given in [**3**]. One proof commonly used is based on non-trivial theorems on the pointwise convergence of Fourier series. A second approach is based on the Euler–MacLaurin summation formula (see [**6**, Section II.10] or [**4**]). Other proofs rely on the product formula for $\sin x$, such as Euler's original proof (see [**6**, pp 62–67] or [**5, 15**]). Yet other proofs involve complex analysis, such as the one in [**12**] or double integrals and Fubini's theorem [**1, 7, 8, 10**]. Without attempting to provide a complete list, there are proofs in [**4, 9, 11, 14**] and references therein.

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Derivation of the result

We start by proving the well-known recursion relations between A_n and A_{n-1}. Using integration by parts and the identity $\sin^2 x = 1 - \cos^2 x$,

$$\begin{aligned} A_n &= \int_0^{\pi/2} \cos x \cos^{2n-1} x\, dx = (2n-1)\int_0^{\pi/2} \sin^2 x \cos^{2(n-1)} x\, dx \\ &= (2n-1)\int_0^{\pi/2} (1-\cos^2 x)\cos^{2(n-1)} x\, dx = (2n-1)(A_{n-1} - A_n). \end{aligned}$$

Hence for $n \geq 1$

$$\int_0^{\pi/2} \sin^2 x \cos^{2(n-1)} x\, dx = \frac{A_n}{2n-1} = \frac{A_{n-1}}{2n}. \tag{3}$$

Next we rewrite A_n in terms of B_n and B_{n-1}. The idea is to use integration by parts twice, introducing the factors x, and then x^2. Using integration by parts a first time we get

$$A_n = \int_0^{\pi/2} 1 \times \cos^{2n} x\, dx = 2n \int_0^{\pi/2} x \sin x \cos^{2n-1} x\, dx.$$

Using integration by parts a second time we get

$$\begin{aligned} A_n &= -n \int_0^{\pi/2} x^2 \big(\cos x \cos^{2n-1} x - (2n-1)\sin^2 x \cos^{2n-2} x\big)\, dx \\ &= -nB_n + n(2n-1)\int_0^{\pi/2} x^2(1-\cos^2 x)\cos^{2(n-1)}\, dx \\ &= (2n-1)nB_{n-1} - 2n^2 B_n. \end{aligned}$$

Hence, for all $n \geq 1$, we have

$$A_n = (2n-1)nB_{n-1} - 2n^2 B_n. \tag{4}$$

This allows us to derive a simple expression for the partial sums of (1). Dividing the identity in (4) by $n^2 A_n$ and then using (3),

$$\frac{1}{n^2} = \frac{(2n-1)B_{n-1}}{nA_n} - \frac{2B_n}{A_n} = \frac{2B_{n-1}}{A_{n-1}} - \frac{2B_n}{A_n}$$

for all $n \geq 1$. Hence we have the telescoping sum

$$\sum_{k=1}^{n} \frac{1}{k^2} = \sum_{k=1}^{n} \left(\frac{2B_{k-1}}{A_{k-1}} - \frac{2B_k}{A_k} \right) = \frac{2B_0}{A_0} - \frac{2B_n}{A_n}$$

for all $n \geq 1$. Now

$$A_0 = \int_0^{\pi/2} 1\, dx = \frac{\pi}{2} \qquad \text{and} \qquad B_0 = \int_0^{\pi/2} x^2\, dx = \frac{\pi^3}{3 \times 8},$$

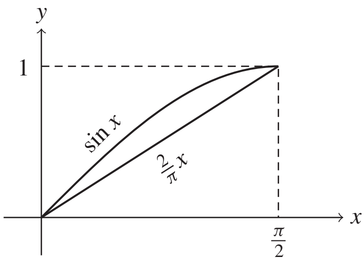
and so

$$\frac{2B_0}{A_0} = \frac{\pi^2}{6}.$$

Hence for all $n \geq 1$ we have

$$\sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6} - 2\frac{B_n}{A_n}. \tag{5}$$

We now estimate B_n in terms of A_n to get a bound for B_n/A_n. The linear function $2x/\pi$ coincides with $\sin x$ for $x = 0$ and for $x = \pi/2$. Because $\sin x$ is concave on $[0, \pi/2]$, we get $\sin x \geq 2x/\pi$ for all $x \in [0, \pi/2]$, as illustrated below.



Using the recursion relation (3) with n replaced by $n + 1$ we get

$$B_n = \int_0^{\pi/2} x^2 \cos^{2n} x \, dx \leq \left(\frac{\pi}{2}\right)^2 \int_0^{\pi/2} \sin^2 x \cos^{2n} x \, dx = \frac{\pi^2}{4} \frac{A_n}{2(n+1)}.$$

Combining the above with (5) we arrive at (2), as required.

We finally note that an induction using (3) and (5) gives Masuoka's representation from [**13**], namely

$$\sum_{k=1}^{n-1} \frac{1}{k^2} = \frac{\pi^2}{6} - \frac{\pi}{4} \frac{(2n)!!}{(2n-1)!!} B_n,$$

but here we have dealt with the error term rather more directly.

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Summary We give a short elementary proof of the well known identity $\zeta(2) = \sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$. The idea is to write the partial sums of the series as a telescoping sum and to estimate the error term. The proof is based on recursion relations between integrals obtained by integration by parts, and simple estimates.

Backwards Induction and a Formula of Ramanujan

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Proof by induction is a marvelously powerful and flexible technique. Unfortunately, the examples of induction typically encountered by students in introductory courses rarely reveal the full range of its versatility. An elegant formula of Ramanujan provides a nice opportunity to use induction in ways that may, at first blush, appear to be *backwards*.

Ramanujan [**1**, p. 323] discovered the remarkable equation

$$\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4 + \cdots}}} = 3.$$

Several proofs have been published; see [**2**, Problem A6] for a terse but complete proof, and [**3**] for a systematic analysis of a whole class of infinitely nested roots, including this one. Here I want to present a short and direct proof of Ramanujan's equation with an amusing property: As I have suggested, each of two critical inequalities is established by a seemingly backwards application of induction.

For $1 \leq n \leq m$, define

$$g(n, m) = \sqrt{1 + n\sqrt{1 + (n+1)\sqrt{\cdots + m}}}.$$

That is, $g(n, m)$ denotes a "middle segment" of the nested root, between the occurrences of n and m. Note that $g(n, m)$ is increasing in m and that $g(n, m) = \sqrt{1 + ng(n+1, m)}$.

CLAIM 1. *For all n, m with $1 \leq n \leq m$, we have $g(n, m) \leq n + 1$.*

For each fixed m, the proof proceeds by downward induction on n from m down to 1.

Base case, $n = m$. In this case $g(n, m) = \sqrt{1 + n}$, which is less than $n + 1$.

Induction step. Assume that $g(n+1, m) \le n+2$, where $n+1 \le m$. Then

$$g(n, m) = \sqrt{1 + ng(n+1, m)} \le \sqrt{1 + n(n+2)} = n+1. \qquad \blacksquare$$

Of course there is nothing invalid in the reasoning of starting at m and counting down to 1. This downward induction could be recast as ordinary induction on k with $n = m - k$ for $k = 0, \ldots, m-1$. The upshot is that for each fixed n, the sequence $g(n, m)$ indexed by m is an increasing sequence bounded by $n+1$. Thus $\lim_{m\to\infty} g(n, m)$ exists, which justifies the definition

$$f(n) = \lim_{m\to\infty} g(n, m) = \sqrt{1 + n\sqrt{1 + (n+1)\sqrt{1 + \cdots}}}$$

for positive integers n. Moreover, $f(n) \le n+1$.

So, our original goal is now to evaluate $f(2)$. Notice that $f(n)$ is increasing in n, and that by definition $f(n) = \sqrt{1 + nf(n+1)}$, which gives us

$$f(n+1) = \frac{f(n)^2 - 1}{n}. \qquad (1)$$

CLAIM 2. *For all n and $k \ge 0$, we have $f(n) > n + 1 - 1/2^k$.*

The proof is by induction on k, for all n simultaneously.

Base case, $k = 0$. Since f is an increasing function,

$$f(n) \le f(n+1) = (f(n)^2 - 1)/n,$$

or $f(n)^2 - nf(n) \ge 1$. Thus $f(n) > n$.

Induction step. Assume $f(n) > n + 1 - 1/2^k$ for all n. By (1),

$$\frac{f(n)^2 - 1}{n} = f(n+1) > n + 2 - \frac{1}{2^k}.$$

Then solving for $f(n)$,

$$\begin{aligned} f(n)^2 &> n^2 + 2n - \frac{1}{2^k}n + 1 \\ &= n^2 + \left(2 - \frac{1}{2^k}\right)n + 1 \\ &> \left(n + \left(1 - \frac{1}{2^{k+1}}\right)\right)^2 \end{aligned}$$

so that $f(n) > n + 1 - 1/2^{k+1}$, as required. $\blacksquare$

True, the induction on k is a perfectly ordinary (upwards!) induction. What can make it feel "backwards" is that we establish facts about $f(n)$ for a particular value of n by appealing to facts previously established about $f(n)$ for larger values of n. For example, $f(2) > 3 - 1/8$ because $f(3) > 4 - 1/4$, which in turn follows from $f(4) > 5 - 1/2$, which in turn follows from $f(5) > 6 - 1$. Formally, this means that the induction had to proceed simultaneously for all n; we could not fix n in advance and carry out the same induction successfully. (In the terminology of mathematical logic, this is an example of what is called a Π_1-induction, which means that the induction formula consists of a universal quantifier—"$\forall n$"—followed by a simple arithmetical statement.)

Claim 2 shows that $f(n)$ is greater than $n+1$ minus any positive number, so $f(n) \geq n+1$. Together with the inequality $f(n) \leq n+1$ that followed from Claim 1, this establishes that $f(n) = n+1$. In particular, $f(2) = 3$, which completes the proof of Ramanujan's formula. So, two complementary and possibly unexpected applications of a versatile technique can be used to prove this striking result.

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Summary Proof by induction is a very versatile technique. A short proof of an elegant formula of Ramanujan involving continued roots provides an opportunity to look at two examples in which induction appears to work backwards.

What Is Special about the Divisors of 24?

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It is a miracle that the human mind can string a thousand arguments together without getting itself into contradictions.

—Eugene Wigner

The divisors of 24 are 1, 2, 3, 4, 6, 8, 12, and 24. Let us pose the following riddle: What is an *interesting* number theoretic characterization of the divisors of 24 among all positive integers?

I will present one characterization in terms of modular multiplication tables. This idea evolved from a question raised by Elliott Mahler in my elementary number theory class. Shortly after introducing the new world of $\mathbb{Z}_n$, I asked my students to write down the multiplication tables for $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}_4$. I then showed them the multiplication table of $\mathbb{Z}_8$ with the intention of drawing their attention to some differences between tables for prime and composite moduli.

$\mathbb{Z}_8$:

*	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

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Upon seeing these tables, Elliott asked, "*I see that 1's in these multiplication tables appear only on the diagonal. Is that always true?*" Of course, looking further we know that this is not always true. For instance, in the table for $\mathbb{Z}_5$, 1 occurs at an off-diagonal position (2, 3) corresponding to the multiplication

$$(2)(3) = 1 \text{ in } \mathbb{Z}_5.$$

Having seen some examples with 1's only on the diagonal and some with 1's also off the diagonal, the following question begs to be answered. *For what values of n do 1's occur only on the diagonal in the multiplication table of $\mathbb{Z}_n$, never off the diagonal?* (Throughout this paper the term "diagonal" refers to the main diagonal.)

I will investigate this question using various tools from number theory and will tie it up with some interesting topics, which seem a priori unrelated to this question. Specifically, the tools used are the Chinese remainder theorem, Dirichlet's theorem on primes in an arithmetic progression, the structure theory of units in $\mathbb{Z}_n$, the Bertrand-Chebyshev theorem, and the extension of the Bertrand-Chebyshev theorem by Erdős and Ramanujan. There is no doubt that some of these tools are rather heavy-duty for the relatively simple question under investigation. My goals are to explore as many interesting topics in number theory as possible via this natural question, and to show the interconnections among these various topics.

The question is answered by the following theorem.

THEOREM. *The multiplication table for $\mathbb{Z}_n$ contains 1's only on the diagonal if and only if n is a divisor of* 24.

Note that the trivial divisor 1 of 24 corresponds to $\mathbb{Z}_1$, which consists of only one element (0). Therefore the requirement of having all ones on the diagonal is vacuously satisfied in this case.

I will give five different arguments. The first three provide a complete proof of the Theorem, while the last two proofs show that the integers n with the diagonal property satisfy $n \leq 24$. The finitely many values of n up to 24 can be dealt with separately to prove the theorem. I will begin with a convenient proposition in the next section.

The diagonal condition

Let me begin by examining the condition "1's in the multiplication table for $\mathbb{Z}_n$ occur only along the diagonal" more closely. For convenience, I will refer to this as the *diagonal condition* or *diagonal property* for n. Let us fix representatives for the elements in $\mathbb{Z}_n$:

$$\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}.$$

Suppose there is a 1 at position (a, b) in the multiplication table for $\mathbb{Z}_n$. This means $ab = 1$ in $\mathbb{Z}_n$. (Then a, and hence also b, is said to be invertible in $\mathbb{Z}_n$.) If the diagonal condition holds for n, then (a, b) has to be a diagonal position. This would mean that $a = b$, and therefore $a^2 = 1$ in $\mathbb{Z}_n$, or equivalently n divides $a^2 - 1$. It is an easy exercise to show that a is invertible in $\mathbb{Z}_n$ if and only if $\gcd(a, n) = 1$. We then have the following proposition.

PROPOSITION. *Let n be a positive integer. Then the following statements are equivalent.*

1. 1*'s in the multiplication table for $\mathbb{Z}_n$ occur only on the diagonal.*
2. *If a is an invertible element in $\mathbb{Z}_n$, then $a^2 = 1$ in $\mathbb{Z}_n$.*

3. *If a is a positive integer that is relatively prime to n, then n divides $a^2 - 1$.*
4. *If p is a prime number that does not divide n, then n divides $p^2 - 1$.*

Proof. In light of the above discussion, the equivalence of the first three statements is clear. Moreover, (4) is a special case of (3). So it is enough to show that (4) implies (3). To this end, let a be a positive integer that is relatively prime to n. If a is 1, then the conclusion is obvious. If $a > 1$, consider the prime factorization $a = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ of a. Since a is relatively prime to n, none of these primes divide n. So by (4), we have $p_i^2 \equiv 1 \mod n$ for all i. Then we have

$$a^2 = (p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k})^2 = (p_1^2)^{a_1} (p_2^2)^{a_2} \cdots (p_k^2)^{a_k} \equiv (1)(1) \cdots (1) \equiv 1 \mod n,$$

as desired. ∎

I will use the equivalent statements of the Proposition interchangeably when referring to integers that have the diagonal property.

The Chinese remainder theorem

In this section I will use the Chinese remainder theorem to give what is probably the shortest proof of the theorem. The Chinese remainder theorem, in its classical form, talks about simultaneous solutions to a system of linear congruences. It can be restated succinctly as an isomorphism of rings [**1**, p. 265]

$$\mathbb{Z}_{ab} \cong \mathbb{Z}_a \oplus \mathbb{Z}_b,$$

whenever a and b are positive integers that are relatively prime. (Multiplication in $\mathbb{Z}_a \oplus \mathbb{Z}_b$ is done component-wise.) Isomorphism of rings means there is a $1:1$ correspondence between $\mathbb{Z}_{ab}$ and $\mathbb{Z}_a \oplus \mathbb{Z}_b$ such that under this correspondence, the addition and multiplication tables in these two rings are the same. As an example, we may verify that $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_6$ under the correspondence $(0, 0) \leftrightarrow 0$, $(0, 1) \leftrightarrow 4$, $(0, 2) \leftrightarrow 2$, $(1, 0) \leftrightarrow 3$, $(1, 1) \leftrightarrow 1$, $(1, 2) \leftrightarrow 5$.

To apply this theorem, we first consider the case of n odd, so that $\gcd(2, n) = 1$. Then, in order for n to have the diagonal property, n has to divide $2^2 - 1 = 3$. This means n has to be either 1 or 3, both of which have the diagonal property. Next, consider the case where n is a power of 2, say $n = 2^t$ for some t, so that $\gcd(3, n) = 1$. As before, for n to have the diagonal property, n has to divide $3^2 - 1 = 8$. It is easily seen that all the divisors of 8 have the diagonal property. Now any positive integer n can be uniquely written as

$$n = 2^t k,$$

where k is odd and t is a nonnegative integer. Then by the Chinese remainder theorem we have the isomorphism

$$\mathbb{Z}_n \cong \mathbb{Z}_{2^t} \oplus \mathbb{Z}_k.$$

From this isomorphism it is easy to see that n has the diagonal property if and only if both 2^t and k have the diagonal property. Combining these pieces, it follows that the only integers with the diagonal property are the divisors of $(8)(3) = 24$.

Dirichlet's theorem on primes in an arithmetic progression

Dirichlet proved the following theorem in 1837, which is a far-reaching extension of Euclid's theorem on the infinitude of primes. It states that, given any two integers s and t that are relatively prime, the arithmetic progression

$$\{sx + t \mid x \text{ is a nonnegative integer}\}$$

contains infinitely many prime numbers; see [**5**, p. 401]. This result is one of the most beautiful results in all of number theory.

Let n be an integer which has the diagonal property. So n has the property that, for any prime p which does not divide n, $n \mid p^2 - 1$. If $n \mid p^2 - 1$, then for every prime divisor q of n, q divides either $p - 1$ or $p + 1$. In other words, every prime p that does not divide n has to be of the form $qx + 1$ or $qx - 1$ for each prime divisor q of n. This is clearly a very strong condition on n. If there is a prime divisor q_0 of n that is bigger than 3, then there will be an arithmetic progression $\{q_0x + r \mid x \geq 0\}$, where $r \neq 0$, 1, or $q_0 - 1$ and $2 \leq r \leq q_0 - 2$. Note that q_0 and r are then relatively prime, and therefore Dirichlet's theorem tells us that this arithmetic progression contains infinitely many primes. In particular, it contains a prime p_0 that does not divide n. This choice of p_0 does not meet the requirement that it is either of the form $q_0x + 1$ or $q_0x - 1$. The upshot is that there is no prime divisor of n bigger than 3, which means that n is of the form 2^u3^v. The smallest prime number that is relatively prime to every number of the form 2^u3^v is 5. Our proposition then tells that n has to divide $5^2 - 1 = 24$, as desired.

We can avoid using the full strength of Dirichlet's theorem. It is enough to assume the special case that the arithmetic progression $5n + 2$ (or $5n + 3$) contains infinitely many primes. This will allow us to show (exactly as above) that 5 cannot divide n. Therefore it follows that n has to divide $5^2 - 1 = 24$. The above proof is, however, more natural. It explains naturally why only primes 2 and 3 can occur in the factorization of n.

The structure theory of units in $\mathbb{Z}_n$

The set of invertible elements (a.k.a. units) in $\mathbb{Z}_n$ is denoted by U_n. This set forms an abelian group under multiplication. The structure of the group U_n has been completely determined. To explain, let $n = p_1^{c_1} p_2^{c_2} \cdots p_k^{c_k}$ be the prime decomposition of n (> 1). The Chinese remainder theorem gives an isomorphism of groups

$$U_n \cong U_{p_1^{c_1}} \oplus U_{p_2^{c_2}} \oplus \cdots \oplus U_{p_k^{c_k}}.$$

It is therefore enough to explain the structure of U_{p^c}. This is given by [**5**, p. 124]:

$$U_{p^c} \cong \begin{cases} C_1 & \text{if } p^c = 2^1 \\ C_2 & \text{if } p^c = 2^2 \\ C_2 \oplus C_{2^{c-2}} & \text{if } p^c = 2^c \text{ and } c \geq 3 \\ C_{\phi(p^c)} & \text{if } p \text{ is odd,} \end{cases}$$

where C_k is the cyclic group of order k, and $\phi(x)$ is Euler's function, which denotes the number of positive integers less than x that are relatively prime to x.

Returning to our problem, recall that from the above proposition, n has the diagonal property if and only if $a^2 = 1$ for all a in U_n. Therefore our job is to simply identify those groups from the above list, which have the property that every element in them has order at most 2. C_1 and C_2 obviously have this property. $C_2 \oplus C_{2^{c-2}}$ will have this property if and only if $c-2 \le 1$, or $c \le 3$. Finally, $C_{\phi(p^c)}$ will have this property for p is odd if and only if $\phi(p^c) = p^{c-1}(p-1) \le 2$. It is easy to see that this last inequality holds only when $p^c = 3$. From these calculations, we note that an integer n with the diagonal property cannot have a prime divisor bigger than 3. Moreover, the maximum power of 3 in n has to be 1, and that of 2 has to be 3. The collection of these integers is given by

$$n = 2^u 3^v, \quad \text{where} \quad 0 \le u \le 3, \quad 0 \le v \le 1,$$

which are exactly the divisors of 24.

The abelian group U_n has a natural $\mathbb{F}_2$-vector space structure precisely when $a^2 = 1$ for all a in U_n. Therefore we can say that n has the diagonal property if and only if U_n is naturally a vector space over $\mathbb{F}_2$.

In the next two sections I will use some results in number theory to show that, if a positive integer n has the diagonal property, then $n \le 24$. The finitely many values of n up to 24 can then be dealt with separately to prove the main theorem.

The Bertrand-Chebyshev theorem

In the year 1845, Bertrand postulated that if $n \ge 2$, then there is always a prime number p such that $n < p < 2n$. Although he did not give a proof, he verified it for all values of n up to three million. A few years later (1852) Chebyshev gave an analytical proof of this result. Elementary proofs, however, had to wait until the next century. In 1919 Ramanujan [**6**] gave the first elementary proof using some properties of the gamma function and the Stirling's formula. His proof could be easily presented without ever mentioning the gamma function. In his first paper in 1932, Erdős [**2**] gave another elementary proof of this theorem using some properties of the binomial coefficients. See [**4**] for a nice presentation of Erdős's proof. Hardy and Wright also use Erdős's proof in their number theory textbook [**3**]. Let us see what this theorem has to say about the question under investigation.

Let n be an integer with the diagonal property. That is, given a prime p that does not divide n, $n \mid p^2 - 1$. Note that if n divides $p^2 - 1$, then $p^2 - 1 \ge n$, or $p \ge \sqrt{n+1}$. Equivalently, looking at the contrapositive, we get the following more appealing statement: If $p < \sqrt{n+1}$, then p divides n.

Here is one of several ways to proceed from this point. Assume that $\sqrt{n+1}/4 \ge 5$ ($\iff n+1 \ge 20^2$) and consider the two intervals

$$\left(\frac{\sqrt{n+1}}{4}, \frac{\sqrt{n+1}}{2}\right), \left(\frac{\sqrt{n+1}}{2}, \sqrt{n+1}\right).$$

By the Bertrand-Chebyshev theorem, each of these intervals has at least one prime. Note that both of these primes are less than $\sqrt{n+1}$. Also, the primes 2, 3, and 5 are less than $\sqrt{n+1}$ because $\sqrt{n+1}/4$ is assumed to be at least 5. Therefore all these primes, and hence their product, divide n. In particular, the product of these primes is at most n. From this we have the following inequality

$$(2)(3)(5)\frac{\sqrt{n+1}}{4}\frac{\sqrt{n+1}}{2} \le n,$$

which simplifies to

$$15(n+1) \le 4n.$$

This is impossible. Therefore we must have $\sqrt{n+1}/4 < 5$, which means $n+1 < 20^2$, or $n \leq 398$. Now we claim that $\sqrt{n+1} \leq 7$. If not, then the product 210 of the primes 2, 3, 5, and 7 would divide n. Since $n \leq 398$, there is only one possibility, namely $n = 210$. But 210 does not have the diagonal property because $(11)(191) = 2101 \equiv 1 \mod 210$. Therefore $\sqrt{n+1} \leq 7$ or $n \leq 48$. Now let us see what happens if $\sqrt{n+1} > 5$. In this case, the primes 2, 3, and 5 divide n. Hence, their product 30 divides n. The only multiple of 30 less than 48 is 30 itself, which does not have the diagonal property because$(13)(7) = 91 \equiv 1 \mod 30$. Therefore $\sqrt{n+1} \leq 5$, which means $n \leq 24$.

The above calculation can be simplified a bit if we use a generalization of the Bertrand-Chebyshev theorem due to Erdős, as we will see in the next section.

Theorems of Erdős and Ramanujan

There are several impressive variations and generalizations of the Bertrand-Chebyshev theorem. A generalization due to Ramanujan [**6**], for instance, says that if $n \geq 6$, then there are at least 2 primes between n and $2n$. This theorem was also proved independently later by Erdős. I will use this theorem to simplify the above proof.

Assume that n has the diagonal property. Then, as above, we have the implication $p < \sqrt{n+1} \implies p \mid n$. Now consider the single interval

$$\left(\frac{\sqrt{n+1}}{2}, \sqrt{n+1} \right).$$

If $\sqrt{n+1}/2 \geq 6$ ($\iff n+1 \geq 144$), this interval has at least two primes by Erdős's theorem. Since $\sqrt{n+1}/2 \geq 6$, the primes 2, 3, and 5 will be less than $\sqrt{n+1}$. Arguing as above, we then have the inequality

$$(2)(3)(5)\left(\frac{\sqrt{n+1}}{2} \right)^2 \leq n,$$

which simplifies to give $30(n+1) \leq 4n$, a contradiction. Therefore, $\sqrt{n+1} < 12$, or $n+1 < 144$. Now we proceed as before by first showing that $\sqrt{n+1} \leq 7$. If not, then the primes 2, 3, 5, and 7, and therefore their product 210, divide n. This is impossible because $n+1 < 144$. Thus $\sqrt{n+1} \leq 7$, which means $n \leq 48$. Similarly if $\sqrt{n+1} > 5$, the primes 2, 3, and 5, and hence also their product 30, divide n. The only multiple of 30 less than 48 is 30 itself, which does not have the diagonal property. Therefore $\sqrt{n+1} \leq 5$, which means $n \leq 24$.

This is only the beginning. There are some further generalizations given by Ramanujan [**6**]. These follow right out of his proof of the Bertrand's postulate. To explain these, let $\pi(x)$ denote the number of primes less than or equal to x. Ramanujan showed that for each positive integer k, there is a prime number p_k such that $\pi(x) - \pi(x/2) \geq k$ if $x \geq p_k$. For example, he showed

$$\pi(x) - \pi(x/2) \geq 1, 2, 3, 4, 5, \ldots \quad \text{if } x \geq 2, 11, 17, 29, 41, \ldots \quad \text{respectively.}$$

The numbers $2, 11, 17, 29, 41, \ldots$ are called the Ramanujan primes. Note that the Bertrand-Chebyshev theorem is covered by the special case

$$\pi(x) - \pi(x/2) \geq 1 \quad \text{if } x \geq 2,$$

and the theorem of Erdős by the case

$$\pi(x) - \pi(x/2) \geq 2 \quad \text{if } x \geq 11.$$

Although we can use these results of Ramanujan to address our question, the bounds thus obtained become worse and it would take more work to get them down to 24.

Further generalisations

Be Wise! Generalise! Instead of working with $\mathbb{Z}_n$, we can look at other rings. For example, we can look at polynomial rings. The question then is: *What are all values of n for which the multiplication table for* $\mathbb{Z}_n[x]$ *has 1's only on the diagonal?*

Similarly, instead of multiplication tables, we can consider multiplication cubes. This is a natural extension of the notion of a multiplication table and is defined similarly. Given a positive integer n, a multiplication cube for $\mathbb{Z}_n$ is a cube $[0, n-1]^3$ whose entry at the coordinate (i, j, k) $(0 \leq i, j, k \leq n-1)$ is the product $ijk \mod n$. Now we can ask the same question for these cubes. *What are all values of n for which the multiplication cube for* $\mathbb{Z}_n$ *has 1's only on the diagonal?*

I invite the reader to venture into these variations and provide as many different proofs of each as he or she can.

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Summary What is an interesting number theoretic characterization of the divisors of 24 among all positive integers? This paper will provide one answer in terms of modular multiplication tables and will give 5 different proofs based on the Chinese remainder theorem, Dirichlet's theorem on primes in an arithmetic progression, the structure theory of units, the Bertrand-Chebyshev theorem, and its generalisations by Erdős and Ramanujan.

Proof Without Words: Runs of Triangular Numbers

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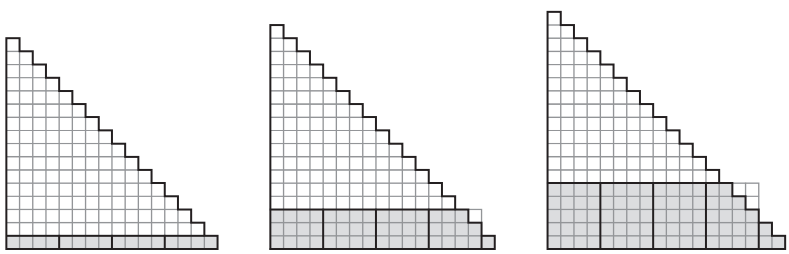
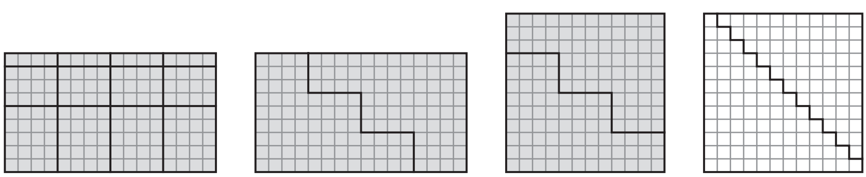
$$T_k = 1 + 2 + \cdots + k \implies \quad T_1 + T_2 + T_3 = T_4$$
$$T_5 + T_6 + T_7 + T_8 = T_9 + T_{10}$$
$$T_{11} + T_{12} + T_{13} + T_{14} + T_{15} = T_{16} + T_{17} + T_{18}$$
$$\vdots$$
$$T_{n^2-n-1} + T_{n^2-n} + \cdots + T_{n^2-1} = T_{n^2} + T_{n^2+1} + \cdots + T_{n^2+n-2}.$$

Proof for $n = 4$.

(i) $T_{16} + T_{17} + T_{18} = T_{15} + T_{14} + T_{13} + 1 \cdot 4^2 + 3 \cdot 4^2 + 5 \cdot 4^2$;



(ii) $(1 + 3 + 5) \cdot 4^2 = 3^2 \cdot 4^2 = (3 \cdot 4)^2 = T_{12} + T_{11}$;



(iii) $\therefore\ T_{11} + T_{12} + T_{13} + T_{14} + T_{15} = T_{16} + T_{17} + T_{18}$.

NOTE: This result complements similar identities for runs of integers [**2**] and runs of squares [**1**]:

$$1 + 2 = 3 \qquad\qquad 3^2 + 4^2 = 5^2$$
$$4 + 5 + 6 = 7 + 8 \qquad\qquad 10^2 + 11^2 + 12^2 = 13^2 + 14^2$$
$$9 + 10 + 11 + 12 = 13 + 14 + 15, \text{ etc.} \qquad 21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2, \text{ etc.}$$

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The Limit Comparison Test Needs Positivity

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The limit comparison test for infinite series appears in almost every modern calculus textbook. One statement is this.

THEOREM 1. *Assume*

(1) $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ *and*

(2) $a_n > 0$ *and* $b_n > 0$ *for all* n.

Then $\sum a_n$ *and* $\sum b_n$ *both converge or both diverge.*

This would be a much more beautiful theorem if we could just drop hypothesis (2). Unfortunately, this is not possible, as the following example illustrates. Let $a_n = \frac{(-1)^n}{\sqrt{n}} + \frac{1}{n}$ and $b_n = \frac{(-1)^n}{\sqrt{n}}$; then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} 1 + \frac{(-1)^n}{\sqrt{n}} = 1$, so hypothesis (1) is true. However, $\sum b_n$ converges and $\sum a_n$ diverges.

In a certain sense, this is the only possible example.

THEOREM 2. *If*

(3) $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ *and*

(4) $\sum b_n$ *converges and* $\sum a_n$ *diverges,*

then $\sum b_n$ *converges conditionally; and if we write* $a_n = b_n + c_n$ *for all* n, *then* $\sum c_n$ *diverges and the* $\{c_n\}$ *are "infinitely smaller" than the* $\{b_n\}$.

Proof. Assume $\sum b_n$ converges absolutely. From (3), $\lim_{n\to\infty} \frac{|a_n|}{|b_n|} = 1$. By the usual limit comparison test for positive series, $\sum a_n$ is absolutely convergent. Consequently, $\sum a_n$ is convergent, a contradiction. So $\sum b_n$ is conditionally convergent. For each n, define $c_n := a_n - b_n$. If $\sum c_n$ converges, then $\sum a_n$ must also, so $\sum c_n$ diverges. Finally, the $\{c_n\}$ are infinitely smaller than the $\{b_n\}$ in the sense that

$$\lim_{n\to\infty} \frac{c_n}{b_n} = \lim_{n\to\infty} \frac{a_n - b_n}{b_n} = \lim_{n\to\infty} \frac{a_n}{b_n} - 1 = 0. \qquad \blacksquare$$

So the most general possible counterexample involves two series, the "big" but convergent $\sum b_n$, and the "little" but divergent $\sum c_n$. Two natural questions are whether for *every* convergent $\sum b_n$ we can find a corresponding $\sum c_n$ to create a counterexample as above, and whether for *every* divergent $\sum c_n$ we can find a corresponding $\sum b_n$ to create a counterexample as above. To be more specific, we ask the following two questions.

(i) Given any convergent series $\sum b_n$, does there exists a "poisoning" series $\sum c_n$ that is small, $c_n = o(b_n)$, and such that $\sum (b_n + c_n)$ is divergent?

(ii) Given any divergent series $\sum c_n$, does there exist a "healing" series $\sum b_n$ that is big, $c_n = o(b_n)$, and such that $\sum b_n$ is convergent?

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As we pointed out above, because of the limit comparison test, in question (i) $\sum b_n$ must be conditionally convergent. In question (ii), the terms of $\sum c_n$ must tend to 0, since otherwise $c_n = o(b_n)$ would be impossible. We will show that these fairly obvious necessary conditions are also sufficient.

For question (i), let $\sum b_n$ be any conditionally convergent series. Define p_n by

$$p_n = \begin{cases} b_n & \text{if } b_n > 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is well known that $\sum p_n = \infty$ [**2**, p. 375]. Next let c_n be nonnegative, equal to zero when p_n is, and satisfy both $c_n = o(p_n)$ and $\sum c_n = \infty$. This can be done since there is no slowest positive divergent series [**1**]. Then $\frac{c_n}{b_n} = 0$ when $p_n = 0$, and $\frac{c_n}{b_n} = \frac{c_n}{p_n}$ when $p_n > 0$, so $c_n = o(b_n)$.

For question (ii), let $\sum c_n$ be any divergent series with terms tending to 0. Let $c_n^* = \sup_{k \geq n} |c_k|$. We will construct $\sum b_n$. Define p_0 to be the first index so that $c_{p_0}^* \leq 4^{-1}$, p_1 to be the first index so that $c_{p_1}^* \leq 4^{-2}$, p_2 to be the first index so that $c_{p_2}^* \leq 4^{-3}, \ldots$. By increasing the p_i as necessary, we may assume that $p_0 - 1$ is a multiple of 2, $p_1 - p_0$ is a multiple of 2^2, $p_2 - p_1$ is a multiple of $2^3, \ldots$. Break the set of indices n such that $1 \leq n < p_0$ into blocks of length 2 and set the values of b_n to be $1, -1$ on each block. Next, break the set of indices n such that $p_0 \leq n < p_1$ into blocks of length 2^2 and set the values of b_n to be $2^{-1}, -2^{-1}, 2^{-1}, -2^{-1}$ on each block. Then, break the set of indices n such that $p_1 \leq n < p_2$ into blocks of length 2^3 and set the values of b_n to be $2^{-2}, -2^{-2}, 2^{-2}, -2^{-2}, 2^{-2}, -2^{-2}, 2^{-2}, -2^{-2}$ on each block. Proceed inductively. For each interval $[p_{i-1}, p_i)$, we have k_i blocks, each of length 2^{i+1}, and

$$\{b_n\}_{n=p_{i-1}}^{p_i - 1} = \Bigg\{ \underbrace{\frac{1}{2^i}, -\frac{1}{2^i}, \frac{1}{2^i}, -\frac{1}{2^i}, \cdots, \frac{1}{2^i}, -\frac{1}{2^i}}_{k_i \cdot 2^{i+1} \text{terms}} \Bigg\}.$$

The sum of the b_n over each of the k_i blocks is 0, while the corresponding sum of the $|b_n|$ is 2. Then $\sum b_n$ converges to 0 and $\sum |b_n| = 2k_1 + 2k_2 + \cdots + 2k_i + \cdots$, so that $\sum b_n$ converges conditionally. For each $i \geq 1$ and each index n such that $p_{i-1} \leq n < p_i$, we have $|c_n| \leq 4^{-i}$ and $|b_n| = 2^{-i}$ so that

$$\left| \frac{c_n}{b_n} \right| \leq \frac{4^{-i}}{2^{-i}} = \frac{1}{2^i}$$

and $c_n = o(b_n)$ as required.

The author was unable to find an instance of this family of examples in the literature. However, this has definitely been in the mathematical folklore for a long time. For example, on page 376 of [**2**], G. H. Hardy, remarks explicitly "...there are no comparison tests for convergence of conditionally convergent series." It seems likely that he had one of these examples in mind to make such a categorical statement.

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Summary The limit comparison test for positive series does not extend to general series. An example is given. In a certain sense, this is the only possible example. Given a conditionally convergent series, there exists a termwise much smaller series so that the sum of the two series diverges. Given a divergent series with terms tending to zero, there exists a convergent but termwise much bigger series.

Möbius Polynomials

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What do bracelets, juggling patterns, and irreducible polynomials over finite fields have in common? We can count all of them using the same techniques. In this paper, we identify a set of polynomials that unify all the counting problems above. Despite their wide application, they have not received an explicit name in the literature. We will christen them *Möbius polynomials* and explore some of their properties. We will highlight the role that Möbius polynomials have played in the contexts above and then use them to give a new combinatorial proof of Euler's totient theorem.

In the first section, we will define our polynomials and derive some key facts about them. After a brief digression to enjoy the graphs of our polynomials in the complex plane, we will see that $M_n(x)$ gives the number of aperiodic bracelets of length n that can be built using x possible types of gems. An immediate corollary will be that $M_n(x) \equiv 0 \pmod{n}$ for all $x \in \mathbb{Z}$. In three subsequent sections, we will apply our polynomials to count juggling patterns, to count irreducible polynomials over finite fields, and to prove Euler's totient theorem.

Definition and properties

To construct our polynomial, we first recall the *Möbius* μ function defined on a positive integer $n = p_1^{e_1} \cdots p_r^{e_r}$, where the p_i are distinct primes:

$$\mu(1) := 1$$

$$\mu\left(n = p_1^{e_1} \cdots p_r^{e_r}\right) := \begin{cases} (-1)^r & \text{if all } e_i = 1 \\ 0 & \text{if any } e_i > 1 \end{cases}$$

Number theorists use μ for *Möbius inversion*, as in [**6**]: If f and g are functions such that

$$\sum_{d|n} f(d) = g(n),$$

then we can solve for f in terms of g via μ:

$$f(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) g(d).$$

We can now meet our main object of study.

DEFINITION. For each integer $n \geq 1$, the nth *Möbius polynomial* is defined to be

$$M_n(x) := \sum_{d|n} \mu\left(\frac{n}{d}\right) x^d.$$

For example, when $n = 12$, we get $M_{12}(x) = x^{12} - x^6 - x^4 + x^2$.

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The first nice property of the Möbius polynomial is that its coefficients are all 0 or ± 1. The coefficient of x^d in $M_n(x)$ is 0 if any square divides $\frac{n}{d}$; therefore, the only nonzero terms that appear in $M_n(x)$ are those corresponding to divisors $d \mid n$ for which $p_1^{e_1-1} \cdots p_r^{e_r-1} \mid d$, where $n = p_1^{e_1} \cdots p_r^{e_r}$. Thus the leading term of $M_n(x)$ is always x^n and the smallest nonzero term is $(-1)^r x^d$, where $d = p_1^{e_1-1} \cdots p_r^{e_r-1}$. The multiplicity of the root at 0 is therefore $p_1^{e_1-1} \cdots p_r^{e_r-1}$.

We note also that $M_n(x)$ always has 2^r nonzero terms, corresponding to the 2^r divisors between $p_1^{e_1-1} \cdots p_r^{e_r-1}$ and n, one for each subset of $\{p_1, \ldots, p_r\}$. If $n > 1$, then for exactly half of these divisors, $\frac{n}{d}$ factors into an odd number of primes, so exactly half of the coefficients are -1 and the other half are 1. This proves that if $n > 1$, then $M_n(1) = 0$.

This is the beginning of an interesting line of study. It turns out that many Möbius polynomials have zeroes at many roots of unity. Here we will just investigate $M_n(-1)$. For $n = 1$ and $n = 2$, we have $M_1(x) = x$ and $M_2(x) = x^2 - x$, so -1 is not a zero, but it is a zero for all higher Möbius polynomials.

THEOREM. *If $n > 2$, then $M_n(-1) = 0$.*

We present two proofs.

Straightforward proof. We will examine each term of $M_n(-1)$ and see that exactly half are negative. There are two cases:

First suppose that $4 \mid n$. As we saw above, the only divisors $d \mid n$ that appear are those for which $p_1^{e_1-1} \cdots p_r^{e_r-1} \mid d$. But all of these divisors are even, so $M_n(x)$ contains only even powers of x. Thus, $M_n(-x) = M_n(x)$, and in particular $M_n(-1) = M_n(1) = 0$.

Next suppose that $4 \nmid n$. Then since $n > 2$, there must be at least one odd prime p dividing n. So among the d for which $p_1^{e_1-1} \cdots p_r^{e_r-1} \mid d$, half contain the final power of that prime p as a factor and half do not. We can pair each d that does not contain that final p with pd, and note that $\mu(\frac{n}{d})(-1)^d$ cancels with $\mu(\frac{n}{pd})(-1)^{pd}$, since $\mu(\frac{n}{d})$ and $\mu(\frac{n}{pd})$ have opposite signs but $(-1)^d$ and $(-1)^{pd}$ have the same sign. For example, in $M_6(x) = x^6 - x^3 - x^2 + x$, we cancel x^6 with $-x^2$ and $-x^3$ with x. Therefore the total sum is 0. ■

Analytic proof. We first note that, since r is the number of *distinct* primes dividing n, we have $n > 2^r$. (The only case when this inequality would not be strict would be if $r = 1$ and $n = 2$, but we excluded that case by hypothesis.) Now, $M_n(-1)$ has 2^r terms, each of which is ± 1, so $|M_n(-1)| \leq 2^r < n$. But we will see in the corollary to our theorem on bracelets, which we will prove independently in a later section, that $M_n(x) \equiv 0 \pmod{n}$ for all $x \in \mathbb{N}$, so $M_n(-1) \equiv M_n(n-1) \equiv 0 \pmod{n}$. Thus, the only possible value of $M_n(-1)$ is 0. ■

We will illustrate these results in the next section before moving on to combinatorial applications.

Digression: The graphs of Möbius polynomials

As mentioned above, many Möbius polynomials have zeroes at many roots of unity. It is interesting to consider a Möbius polynomial as a function of a complex argument z and to examine its values when z lies on the unit circle in $\mathbb{C}$. We can graph them by evaluating $M_n(z)$ for $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$, and plotting the results in the complex plane. For example, FIGURE 1 shows the graphs of $M_{15}(z) = z^{15} - z^5 - z^3 + z$ and $M_{17}(z) = z^{17} - z$.

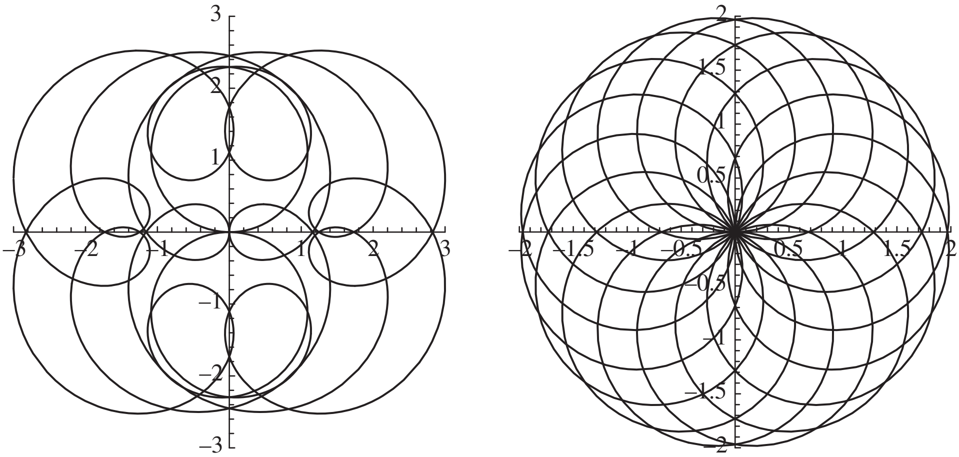

Figure 1 The graphs of $M_{15}(z)$ and $M_{17}(z)$ for $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$

The graphs cross the origin at zeroes of the polynomials. $M_{15}(z)$ has zeroes at ± 1 and $\pm i$, while $M_{17}(z) = z^{17} - z = z(z^{16} - 1)$ has zeroes at all sixteenth roots of unity.

The symmetries of the graphs reflect the structure of the polynomials. $M_{15}(z)$ satisfies $M_{15}(-z) = -M_{15}(z)$, giving the graph rotational symmetry around the origin; $M_{15}(\bar{z}) = \overline{M_{15}(z)}$, giving the graph vertical symmetry across the real axis; and $M_{15}(-\bar{z}) = -\overline{M_{15}(z)}$, giving the graph horizontal symmetry across the imaginary axis. $M_{17}(z)$, on the other hand, satisfies

$$M_{17}(\omega z) = \omega z\big((\omega z)^{16} - 1\big) = \omega z(z^{16} - 1) = \omega M_{17}(z),$$

for the sixteenth root of unity $\omega = e^{2\pi i/8}$, so the graph is symmetric with respect to rotation through ω.

There are many more beautiful symmetries and patterns to be discovered by graphing Möbius polynomials and, more generally, other functions $f : \mathbb{C} \to \mathbb{C}$ on the unit circle. I invite you to play with them yourself. In the meantime, we will return to combinatorial applications of Möbius polynomials.

Möbius polynomials and bracelets

We are ready for our first combinatorial result on Möbius polynomials. We would like to build circular bracelets of length n using x possible types of gems. We can think of each bracelet as a word of n letters, and we have x choices for each letter, so there are x^n possible bracelets in all. However, we wish to exclude those bracelets that are *periodic* with respect to any proper divisor $d \mid n$, that is, those that after rotating by d gems look the same as themselves. For example, the bracelets XOXOXO and OXOOXO are periodic with periods 2 and 3, respectively, but the bracelet XOXOOO is aperiodic.

THEOREM. *The Möbius polynomial $M_n(x)$ gives the number of aperiodic bracelets of length n with x possible types of gems.*

Proof. Note that every bracelet of length n is periodic with respect to some *fundamental* (that is, shortest) period $d \mid n$; the aperiodic bracelets are those for which $d = n$. For each $d \mid n$, there is a one-to-one correspondence between the bracelets of length n with fundamental period d and the aperiodic bracelets of length d. Let us define $g(d)$ to be the number of aperiodic bracelets of length d. Then our correspondence gives us $\sum_{d|n} g(d) = x^d$. We now apply Möbius inversion:

$$g(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) x^d = M_n(x).$$

This proves the theorem. ■

COROLLARY. *The Möbius polynomial satisfies* $M_n(x) \equiv 0 \pmod{n}$ *for all* $x \in \mathbb{N}$.

Proof. We can sort the aperiodic bracelets of length n into groups that look like each other after rotation. For example, with $n = 6$ and $x = 2$, one group would be {XOXOOO, OXOXOO, OOXOXO, OOOXOX, XOOOXO, OXOOOX}. Because we are considering only aperiodic bracelets, each group contains exactly n bracelets, proving the corollary. ■

We will recycle this argument for several applications in the following sections, first to count juggling patterns and irreducible polynomials, and then to prove Euler's totient theorem. We note here that Bender and Goldman [**2**] derive a similar formula in which they count the total number of rotationally distinct bracelets of length n (including periodic ones), meaning for example that they consider OXOOXO and OOXOOX to be the same. Thus each divisor $d \mid n$ contributes

$$\frac{1}{d} M_d(x) = \frac{1}{d} \sum_{c|d} \mu\left(\frac{d}{c}\right) x^c$$

bracelets, giving a total of

$$\sum_{d|n} \frac{1}{d} \sum_{c|d} \mu\left(\frac{d}{c}\right) x^c$$

bracelets in all. This expression simplifies pleasantly when we use Gauss's identity $n = \sum_{d|n} \phi(n)$, giving

$$\frac{1}{n} \sum_{d|n} \phi\left(\frac{n}{d}\right) x^d.$$

We omit the details because we will not need them here.

Möbius polynomials and juggling patterns

Jugglers and mathematicians describe juggling patterns using *siteswap* notation, in which strings of nonnegative integers represent throws to different heights. For example, 441 represents throwing two balls high and then quickly passing a third ball from one hand to the other underneath them. (Actually, the hand order, and even the number of hands, are irrelevant to the notation. Each positive integer just represents the number of time beats from when a ball is thrown to when it is thrown again, and a zero represents a beat in which no ball is thrown.) Jugglers usually repeat a sequence of throws periodically, so 441 is shorthand for a pattern in which one would throw red, blue, and green balls in the order R G B B R G G B R R G B B R G G B R. . . .

Many websites animate juggling patterns, notably Boyce's Juggling Lab [**5**]. Siteswap notation has spread like wildfire from academia to the mainstream juggling community of hobbyists, performers, and competitors. In 2005, for example, Japanese street performer Kazuhiro Shindo won the International Jugglers' Association championships with a routine based on variations on 7441, and he even shaved the formula into the back of his head (FIGURE 2).

Figure 2 Kazuhiro Shindo and 7441 (photo courtesy of Joyce Howard)

Academically-minded jugglers have long known that a string of nonnegative integers $a_1 \cdots a_n$ is a valid siteswap pattern if and only if for all $1 \le i \ne j \le n$, we have $i + a_i \not\equiv j + a_j \pmod{n}$. This condition ensures that the balls do not collide upon landing, since throw i lands at time $i + a_i \pmod{n}$. In their seminal paper [**4**], Buhler, Eisenbud, Graham, and Wright prove (via some nontrivial combinatorics) the remarkable theorem that the number of patterns of period n with strictly fewer than b balls is exactly b^n.

As an example, the predicted $2^4 = 16$ patterns of period four with zero or one ball(s) are 0000, 4000, 0400, 0040, 0004, 3001, 1300, 0130, 0013, 2020, 0202, 1111, 2011, 1201, 1120, and 0112. Actual jugglers, however, would only list 4000, 3001, and 2011, since all the others are either cyclic copies of these three (recall that a juggler repeats a pattern indefinitely without caring where it begins and ends) or have fundamental period less than four. For example, 2020 has fundamental period two.

A juggler would define $f(n, b)$ to be the number of siteswap patterns of fundamental period n, where cyclic copies of patterns such as 3001 and 0130 are considered the same. Then for each divisor $d \mid n$, the total of b^n includes patterns of fundamental period d, and each pattern is counted d times because it can be rotated d places until it repeats itself. For example, the period two pattern 20 is counted twice in the list of period four patterns above. The total thus breaks down as follows:

$$\sum_{d|n} d f(d, b) = b^n.$$

Let us define the temporary function $T(d, b) := d f(d, b)$. Then our formula becomes $\sum_{d|n} T(d, b) = b^n$, and Möbius inversion gives us

$$T(n, b) = \sum_{d|n} \mu\left(\frac{n}{d}\right) b^d = M_n(b),$$

that is,

$$f(n, b) = \frac{1}{n} M_n(b).$$

For example, the count of patterns of period four with zero or one ball(s) is

$$\begin{aligned}\frac{1}{4}M_4(2) &= \frac{1}{4}\sum_{d|4}\mu\left(\frac{4}{d}\right)2^d \\ &= \frac{1}{4}\big[\underbrace{\mu(4)2^1}_{d=1} + \underbrace{\mu(2)2^2}_{d=2} + \underbrace{\mu(1)2^4}_{d=4}\big] \\ &= \frac{1}{4}[0 - 4 + 16] = 3,\end{aligned}$$

confirming the three distinct patterns 4000, 3001, and 2011.

The original four authors, all accomplished jugglers themselves, gave this formula in [**4**]. In fact, they derived the formula for the number of patterns with *exactly b* balls:

$$\begin{aligned}f(n, b+1) - f(n, b) &= \frac{1}{n}M_n(b+1) - \frac{1}{n}M_n(b) \\ &= \frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)\big[(b+1)^d - b^d\big],\end{aligned}$$

(This expression is denoted $M(n, b)$ in [**4**]; we will avoid this notation because it could be confused with our $M_n(b)$.)

For example, the number of period-three patterns with exactly three balls is

$$\frac{1}{3}\big[\underbrace{\mu(3)(4^1 - 3^1)}_{d=1} + \underbrace{\mu(1)(4^3 - 3^3)}_{d=3}\big] = 12,$$

and the patterns are 423, 441, 450, 522, 531, 603, 612, 630, 711, 720, 801, and 900.

Möbius polynomials and irreducible polynomials

Möbius polynomials find another application in counting irreducible polynomials over finite fields. In fact, with a little finite field theory, we can give a formula using the same proof as for juggling patterns.

THEOREM. *The number of monic irreducible polynomials of degree exactly n over the finite field* $\mathbb{F}_p$ *is*

$$\frac{1}{n}M_n(p) = \frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)p^d.$$

Proof. Define $f(n)$ to be the number of such polynomials. Each element $\alpha \in \mathbb{F}_{p^n}$ satisfies an irreducible monic polynomial of some degree d over $\mathbb{F}_p$, and we know $d \mid n$ because $\mathbb{F}_p(\alpha) = \mathbb{F}_{p^d} \subseteq \mathbb{F}_{p^n}$. In this list of irreducible polynomials, each irreducible polynomial of degree d is counted d times, once for each of its roots. Therefore, $\sum_{d|n} df(d) = p^n$, and just as in the derivation of the juggling formula, we get $f(n) = \frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)p^d = \frac{1}{n}M_n(p)$. ■

In fact, the same proof gives a more general version of the theorem, that the number of monic irreducible polynomials of degree exactly n over the finite field $\mathbb{F}_{p^e}$ is

$$\frac{1}{n}M_n(p^e) = \frac{1}{n}\sum_{d|n}\mu\left(\frac{n}{d}\right)p^{de}.$$

This formula, of course, is known in the literature, for example in Dornhoff and Hohn [**9**]. In [**8**], Chebolu and Mináč give a proof based on inclusion-exclusion that interprets each term in the sum above in terms of field theory.

Möbius polynomials and Euler's totient theorem

Our corollary on bracelets gives us an immediate combinatorial proof of Fermat's little theorem. Take $n = p$ to be prime, and we get $M_p(x) = x^p - x \equiv 0 \pmod{p}$. This proof of Fermat appeared first in Dickson [**7**] and later with many interesting variations in Anderson, Benjamin, and Rouse [**1**]. In their master compendium of combinatorial proofs [**3**], Benjamin and Quinn issue the challenge: "Although we do not know of a combinatorial proof of [Euler's totient theorem that $a^{\phi(n)} \equiv 1 \pmod{n}$ when a and n are relatively prime], we would love to see one!" We can use our corollary to prove a special case of Euler, when $n = p^e$. In this case, Euler's theorem becomes $a^{p^e - p^{e-1}} \equiv 1 \pmod{p^e}$, or $a^{p^e} \equiv a^{p^{e-1}} \pmod{p^e}$.

Proof of Euler's totient theorem (special case). We evaluate the Möbius polynomial $M_{p^e}(x)$ at $x = a$:

$$M_{p^e}(a) := \sum_{d|n} \mu\left(\frac{p^e}{d}\right) a^d$$

This polynomial has only two nonzero terms, which are $a^{p^e} - a^{p^{e-1}}$. By the corollary on bracelets, we have $p^e \mid (a^{p^e} - a^{p^{e-1}})$, proving the special case. ∎

It is easy to derive the general version of Euler from the combinatorial special case as follows: Suppose a and $n = p_1^{e_1} \cdots p_r^{e_r}$ are relatively prime, where the p_i are distinct primes. Then for each prime p_i, we have:

$$\begin{aligned}
& p_i^{e_i} \,\Big|\, \left(a^{p_i^{e_i}} - a^{p_i^{e_i-1}}\right) && \text{by the combinatorial special case} \\
& p_i^{e_i} \,\Big|\, \left[a^{p_i^{e_i-1}}\left(a^{p_i^{e_i} - p_i^{e_i-1}} - 1\right)\right] && \text{by factoring} \\
& p_i^{e_i} \,\Big|\, \left(a^{p_i^{e} - p_i^{e_i-1}} - 1\right) && \text{since } (a, p_i) = 1 \\
& p_i^{e_i} \,\Big|\, \left(a^{\phi(p_i^{e_i})} - 1\right) &&
\end{aligned}$$

Since ϕ is multiplicative on relatively prime arguments, we have

$$\phi(n) = \phi(p_1^{e_1}) \cdots \phi(p_r^{e_r}).$$

In particular, each $\phi(p_i^{e_i}) \mid \phi(n)$, so

$$\left(a^{\phi(p_i^{e_i})} - 1\right) \,\Big|\, \left(a^{\phi(n)} - 1\right),$$

and transitivity yields $p_i^{e_i} \mid (a^{\phi(n)} - 1)$. Combining these gives us $n \mid (a^{\phi(n)} - 1)$, the general version of Euler's theorem.

Acknowledgments I thank Art Benjamin and Robert Mena for suggesting this investigation and many useful subsequent thoughts. The referees also offered many helpful ideas that dramatically improved this paper.

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Summary We introduce the Möbius polynomial $M_n(x) = \sum_{d|n} \mu\left(\frac{n}{d}\right) x^d$, which gives the number of aperiodic bracelets of length n with x possible types of gems, and therefore satisfies $M_n(x) \equiv 0 \pmod{n}$ for all $x \in \mathbb{Z}$. We derive some key properties, analyze graphs in the complex plane, and then apply Möbius polynomials combinatorially to juggling patterns, irreducible polynomials over finite fields, and Euler's totient theorem.

To appear in *College Mathematics Journal*, January 2013

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PROBLEMS

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PROPOSALS

To be considered for publication, solutions should be received by May 1, 2013.

1906. *Proposed by Yagub N. Aliyev, Department of Mathematics, Qafqaz University, Khyrdalan, Azerbaijan.*

Let ABC be an equilateral triangle and X a point in its interior such that $\angle CXA = 2\pi/3$. Suppose that the lines AX and BX intersect $\overline{BC}$ and $\overline{AC}$ at D and E, respectively. Prove that

$$\frac{1}{[\text{Area}(ABD)]^2} + \frac{1}{[\text{Area}(ADC)]^2} = \frac{1}{[\text{Area}(BDE)]^2 + [\text{Area}(ADE)]^2}.$$

1907. *Proposed by Daniel Lopez Aguayo, student, Institute of Mathematics, UNAM, Morelia, Mexico.*

Let R be a commutative ring with 1 such that there exists an integer n, $n \geq 2$, with the property that $r^n = r$ for all $r \in R$. Find with proof the intersection of all maximal ideals of R.

1908. *Proposed by D. M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania.*

Let ABC be an arbitrary triangle with $a = BC$, $b = AC$, and $c = AB$. Denote by s, r, and R, the semiperimeter, inradius, and circumradius of $\triangle ABC$, respectively. Prove that if m and n are positive real numbers, then the following inequalities hold:

Math. Mag. **85** (2012) 384–391. doi:10.4169/math.mag.85.5.384. © Mathematical Association of America

We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must, in general, be accompanied by solutions and by any bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

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(a) $\dfrac{a}{mb+nc}+\dfrac{b}{mc+na}+\dfrac{c}{ma+nb}\ge\dfrac{4s^2}{(m+n)(s^2+r^2+4Rr)}$,

(b) $\dfrac{1}{(ma+nb)^2}+\dfrac{1}{(mb+nc)^2}+\dfrac{1}{(mc+na)^2}\ge\dfrac{27}{4(m+n)^2s^2}$.

1909. *Proposed by Howard Cary Morris, STCC, Memphis, TN.*

Let A and B be two countably infinite subsets of the interval $(0, 1)$.

(a) Show that if A and B are dense in $(0, 1)$, then there is a bijective function $f : A \to B$ such that f is strictly increasing on A, i.e., for every $a_1, a_2 \in A$, $f(a_1) < f(a_2)$ whenever $a_1 < a_2$.

(b) Show that such a function does not necessarily exist if A and B are not dense in $(0, 1)$.

1910. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Let $f : [a, b] \to [0, \infty)$ be a continuous function. Show that the following limit exists and calculate its value:

$$\lim_{n\to\infty} n\left[\sqrt[n]{\int_a^b (f(x))^{n+1}dx}-\sqrt[n]{\int_a^b (f(x))^{n}dx}\right].$$

Quickies

Answers to the Quickies are on page 391.

Q1025. *Proposed by Michael W. Botsko, Saint Vincent College, Latrobe, PA.*

Determine all real-valued continuous functions on $\mathbb{R}$ such that

(a) $f(x+y)=f(x)+f(y)+e^xe^y$ for all $x, y \in \mathbb{R}$,

(b) $f(x+y)=f(x)+f(y)+(e^x-1)(e^y-1)$ for all $x, y \in \mathbb{R}$.

Q1026. *Proposed by Tom Moore, Department of Mathematics and Computer Science, Bridgewater State University, Bridgewater, MA.*

Let R be a ring with unity $1_R \ne 0_R$. Can R have an odd number of idempotents?

Solutions

A corollary of Karamata's inequality **December 2011**

1881. *Proposed by Slavko Simic, Mathematical Institute SANU, Belgrade, Serbia.*

Let $a < b$ be real numbers and $f : [a, b] \to \mathbb{R}$ a convex function (equivalently a concave upward function). Define

$$F(s,t)=f(s)+f(t)-2f\left(\frac{s+t}{2}\right).$$

Prove that $F(s, t) \le F(a, b)$ for every $s, t \in [a, b]$.

Solution by Bruce S. Burdick, Roger Williams University, Bristol, RI.
We prove the following stronger fact about F. Whenever $s \leq x \leq t$ with s, x, and $t \in [a,b]$, we have that

$$F(s,x) + F(x,t) \leq F(s,t). \tag{1}$$

Recall that, by definition, f is convex in $[a,b]$ if and only if $f(\lambda s + (1-\lambda)t) \leq \lambda f(s) + (1-\lambda) f(t)$ for every $\lambda \in [0,1]$ and $s, t \in [a,b]$. Because F takes non-negative values ($\lambda = \frac{1}{2}$ in the above definition of convexity) and $F(s,t) = F(t,s)$, the claim in the problem follows from inequality (1) by observing that for $s, t \in [a,b]$ with $s \leq t$,

$$F(a,b) \geq F(a,s) + F(s,b) \geq F(a,s) + F(s,t) + F(t,b) \geq F(s,t).$$

To prove inequality (1), consider $u, v, \alpha, \beta \in [a,b]$ such that $u + v = \alpha + \beta$ and $\alpha \leq u \leq v \leq \beta$. By convexity, if $\lambda = (\beta - u)/(\beta - \alpha)$ and $\mu = (\beta - v)(\beta - \alpha)$, then

$$f(u) = f(\lambda\alpha + (1-\lambda)\beta) \leq \lambda f(\alpha) + (1-\lambda) f(\beta) \text{ and}$$
$$f(v) = f(\mu\alpha + (1-\mu)\beta) \leq \mu f(\alpha) + (1-\mu) f(\beta).$$

Since $\lambda + \mu = 1$, adding the above inequalities gives

$$f(u) + f(v) \leq f(\alpha) + f(\beta).$$

Therefore, for $s \leq x \leq t$,

$$f\left(\frac{s+x}{2}\right) + f\left(\frac{x+t}{2}\right) \geq f(x) + f\left(\frac{s+t}{2}\right),$$

and then

$$\begin{aligned} &F(s,t) - F(s,x) - F(x,t) \\ &\quad = 2f\left(\frac{s+x}{2}\right) + 2f\left(\frac{x+t}{2}\right) - 2f(x) - 2f\left(\frac{s+t}{2}\right) \geq 0. \end{aligned}$$

Editor's Note. Many of our solvers noticed that this problem can be solved easily by using Karamata's inequality: For $n \geq 2$, if $\{x_i\}_{i=1..n}$, $\{y_i\}_{i=1..n}$ in $[a,b]$ are two sequences of real numbers such that $\sum_{k=1}^{i} x_k \leq \sum_{k=1}^{i} y_k$, $i = 1, 2, \ldots, n-1$, and $\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} y_k$, and f is a convex function as in our problem, then

$$\sum_{k=1}^{n} f(x_k) \leq \sum_{k=1}^{n} f(y_k).$$

The proof included here uses basically Karamata's inequality for $n = 2$. Elias Lampakis pointed out that this problem is a particular case of a result in [V. Cirtoaje, The Best Upper Bound for Jensen's Inequality, *The Australian Journal of Mathematical Analysis and Applications* **7** (2011), Article 22], which states that, for a sequence of numbers $\{x_i\}_{i=1..n}$ in $[a,b]$ and f as before,

$$\frac{1}{n}\sum_{k=1}^{n} f(x_k) - f\left(\frac{1}{n}\sum_{k=1}^{n} f(x_k)\right) \leq \left(1 - \frac{1}{n}\right)\left(f(a) + f(b) - 2f\left(\frac{a+b}{2}\right)\right).$$

Using Karamata's inequality, Marian Tetiva proved that the result of the problem is true if F is replaced by $F(s,t) = \lambda f(s) + (1-\lambda) f(t) - f(\lambda s + (1-\lambda)t)$ for an arbitrary $\lambda \in [0, 1]$. Alfred Witkowski observed that this problem already appeared in the *Problem Corner of the Research Group in Mathematical Inequalities and Applications* (http://www.rgmia.org/pc.php) as Problem 2 (2010).

Also solved by George Apostolopulous (Greece), Robert Calcaterra, Carol A. Downes, "Fejéntaláltuka Szeged" problem solving group (Hungary), Eugene A. Herman, Enkel Hysnelaj (Australia) and Elton Bojaxhiu (Germany), The Iowa State University Undergraduate Problem Solving Group, Omran Kouba (Syria), Elias Lampakis (Greece), Peter McPolin (Northern Ireland), Paolo Perfetti (Italy), Ángel Plaza (Spain), Joel Schlosberg, Marian Tetiva (Romania), Traian Viteam (Germany), Alfred Witkowski (Poland), and the proposer. There was one incomplete solution and seven solutions that assumed differentiability.

A telescoping series of arctangents **December 2011**

1882. *Proposed by Timothy Hall, PQI Consulting, Cambridge, MA.*

Prove that

$$\sum_{n=1}^{\infty} \arctan \frac{2}{n^2} = \frac{3}{4}\pi$$

and find an error estimate for the partial sums.

I. *Solution by Michel Bataille, Rouen, France.*
Recall that for $ab < 1$, the following formula holds:

$$\arctan(a) + \arctan(b) = \arctan\left(\frac{a+b}{1-ab}\right). \tag{1}$$

For $n > 1$, let $a = 1/(n-1)$ and $b = -1/(n+1)$. It follows that

$$\arctan\left(\frac{1}{n-1}\right) - \arctan\left(\frac{1}{n+1}\right) = \arctan\frac{2}{n^2}.$$

Thus, for every integer $N > 2$,

$$\begin{aligned}\sum_{n=1}^{N} \arctan\frac{2}{n^2} &= \arctan(2) + \sum_{n=2}^{N}\left(\arctan\left(\frac{1}{n-1}\right) - \arctan\left(\frac{1}{n+1}\right)\right)\\ &= \arctan(2) + \arctan(1) + \arctan\frac{1}{2} - \arctan\frac{1}{N} - \arctan\frac{1}{N+1},\end{aligned}$$

and since $\arctan(1) = \pi/4$ and $\arctan(2) + \arctan\frac{1}{2} = \pi/2$, then

$$\sum_{n=1}^{N} \arctan\frac{2}{n^2} = \frac{3}{4}\pi - \arctan\frac{1}{N} - \arctan\frac{1}{N+1}.$$

Letting $N \to \infty$ yields

$$\sum_{n=1}^{\infty} \arctan\frac{2}{n^2} = \frac{3}{4}\pi,$$

and since $\arctan(x) < x$ for positive x,

$$0 < \frac{3}{4}\pi - \sum_{n=1}^{N} \arctan\frac{2}{n^2} = \arctan\frac{1}{N} + \arctan\frac{1}{N+1} < \frac{2}{N}.$$

Thus, the partial sum $\sum_{n=1}^{N}\arctan(2/n^2)$ approaches $\frac{3}{4}\pi$ from below with an error less than $2/N$.

II. *Solution by Bruce S. Burdick, Roger Williams University, Bristol, RI.*
For every complex z, let $\arg(z)$ be the argument of z in the range $[-\pi, \pi]$. Note that $\arctan(x) < |x|$, and thus for every $N \geq 1$,

$$0 < \sum_{n=N}^{\infty}\arctan\frac{2}{n^2} < \int_{n=N}^{\infty}\frac{2}{x^2}dx = \frac{2}{N}.$$

For $N = 1$, this inequality implies that the required sum is positive and less than 2, and thus it can be written as the argument of a product of complex numbers as follows:

$$\sum_{n=1}^{\infty}\arctan\frac{2}{n^2} = \arg\prod_{n=1}^{\infty}\left(1+\frac{2}{n^2}i\right).$$

It is well known that

$$\frac{1}{\pi z}\sin(\pi z) = \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right).$$

Letting $z = 1 - i$ gives

$$\sum_{n=1}^{\infty}\arctan\frac{2}{n^2} = \arg\left(\frac{1}{\pi(1-i)}\sin(\pi(1-i))\right) = \arg\left(\frac{1+i}{2\pi}\sin(\pi i)\right)$$
$$= \arg\left((-1+i)\frac{\sinh(\pi)}{2\pi}\right) = \frac{3}{4}\pi,$$

and as established before, the error term $\sum_{n=N}^{\infty}\arctan(2/n^2)$ is positive and bounded above by $2/N$.

Editor's Note. Many solution attempts incorrectly stated that (1) is true for any $ab \neq 1$. In fact, if $ab > 1$, then $\arctan(a) + \arctan(b) = \pi + \arctan((a+b)/(1-ab))$. Other attempts were deemed incomplete for not checking the domain of x where $\arctan(\tan x) = x$ is valid. This problem has appeared many times in print before. Some of the references mentioned are Gem 34, in D. D. Bonar and M. J. Khoury, *Real Infinite Series*, MAA, 2006, pp. 106, and Problem 399, *College Mathematics Journal*, **21** (1990), 253–254. The editor's note to Problem 399 contains further references to earlier publications and generalizations.

Also solved by George Apostolopulous (Greece), Elton Bojaxhiu (Germany) and Enkel Hysnelaj (Australia), Tim Cross (United Kingdom), Paul Deiermann, David Doster, Dmitry Fleischman, Michael Goldenberg and Mark Kaplan, Eugene A. Herman, John C. Kieffer, Omran Kouba (Syria), Harris Kwong, Elias Lampakis (Greece), Kee-Wai Lau (China), László Lipták, Northwestern University Math Problem Solving Group, Moubinool Omarjee (France), Paolo Perfetti (Italy), Ángel Plaza (Spain), Nicholas C. Singer, Traian Viteam (Germany), Albert R. Whitcomb, John Zacharias, and the proposer. There were 13 incomplete or incorrect solutions.

Ptolemy's Inequality for hexagons **December 2012**

1883. *Proposed by Panagiote Ligouras, "Leonardo da Vinci" High School, Noci, Italy.*

The lengths of the sides of a plane hexagon $ABCDEF$ (not necessarily convex) satisfy that $2AB = BC$, $2CD = DE$, and $2EF = FA$. Prove that

$$\frac{FA}{FC}+\frac{BC}{BE}+\frac{DE}{DA} \geq 2.$$

Solution by Michael Vowe, Therwil, Switzerland.
Let $x = AC$, $y = AE$, and $z = CE$. Ptolemy's Inequality applied to the quadrilaterals $ABCE$, $ACED$, and $ACEF$, together with the fact that $2AB = BC$, $2CD = DE$, and $2EF = FA$, imply that

$$\begin{aligned} AB(z + 2y) &= AB \cdot z + BC \cdot y \geq BE \cdot x, \\ CD(y + 2x) &= CD \cdot y + DE \cdot x \geq DA \cdot z, \text{ and} \\ EF(x + 2z) &= EF \cdot x + FA \cdot z \geq FC \cdot y. \end{aligned}$$

Thus

$$\frac{FA}{FC} + \frac{BC}{BE} + \frac{DE}{DA} \geq 2\left(\frac{EF}{FC} + \frac{AB}{BE} + \frac{CD}{DA}\right) \geq \frac{2y}{x + 2z} + \frac{2x}{z + 2y} + \frac{2z}{y + 2x}.$$

Let $9u = x + 2z$, $9v = y + 2x$, and $9w = z + 2y$, so that $x = 4v - 2w + u$, $y = 4w - 2u + v$, and $z = 4u - 2v + w$. By the Arithmetic Mean–Geometric Mean inequality, it follows that

$$\begin{aligned} \frac{2y}{x + 2z} + \frac{2x}{z + 2y} + \frac{2z}{y + 2x} &= \frac{8w - 4u + 2v}{9u} + \frac{8v - 4w + 2u}{9w} + \frac{8u - 4v + 2w}{9v} \\ &= \frac{2}{3}\left(\frac{u}{v} + \frac{v}{w} + \frac{w}{u}\right) + \frac{2}{9}\left(\frac{u}{v} + \frac{v}{u}\right) + \frac{2}{9}\left(\frac{v}{w} + \frac{w}{v}\right) + \frac{2}{9}\left(\frac{w}{u} + \frac{u}{w}\right) - \frac{4}{3} \\ &\geq 2\sqrt[3]{\frac{u}{v} \cdot \frac{v}{w} \cdot \frac{w}{u}} + \frac{4}{9}\sqrt{\frac{u}{v} \cdot \frac{v}{u}} + \frac{4}{9}\sqrt{\frac{v}{w} \cdot \frac{w}{v}} + \frac{4}{9}\sqrt{\frac{w}{u} \cdot \frac{u}{w}} - \frac{4}{3} \\ &= 2 + \frac{4}{9} + \frac{4}{9} + \frac{4}{9} - \frac{4}{3} = 2. \end{aligned}$$

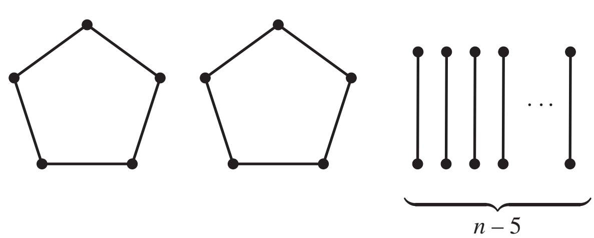
Equality occurs if and only if the quadrilaterals $ABCE$, $ACED$, and $ACEF$ are cyclic and $u = v = w$ (equivalently, $x = y = z$); that is, if and only if $ABCDEF$ is a cyclic hexagon such that $2AB = 2CD = 2EF = BC = DE = FA$.

Also solved by George Apostolopulous (Greece), Michel Bataille (France), Elton Bojaxhiu (Germany) and Enkel Hysnelaj (Australia), Minh Can, Elias Lampakis (Greece), and the proposer.

Triangle-free graphs with small independence number **December 2012**

1884. *Proposed by Khodakhast Bibak, Department of Combinatorics & Optimization, University of Waterloo, Waterloo, Canada.*

Let $n \geq 5$ be a positive integer. Consider the graphs G on $2n$ vertices that are triangle-free and have independence number $\alpha(G) < n$. Prove that if G has the minimum possible number of edges, then G is isomorphic to the disjoint union of two cycles of length 5 and $n - 5$ edges.



Solution by László Lipták, Oakland University, Rochester, MI.

Consider a graph G satisfying the conditions of the problem with a minimum number of edges. Clearly G is simple, since deleting a parallel edge or loop does not change the independence number. Deleting any edge of G can increase the independence number by at most 1, so the minimality of the number of edges implies that $\alpha(G) = n - 1$ and deleting any edge of G increases the independence number by exactly 1 (such a graph is called α-*critical*).

First assume that vw is an edge in G such that the degree of v is 1. If w is incident to another edge wz (so $z \neq v$), then there must be an independent set W in $G - wz$ of size n. Clearly W contains w and z, hence it does not contain v. Thus $(W \cup \{v\}) \setminus \{w\}$ is also an independent set in G of size n, which is a contradiction. So a vertex of degree 1 can occur in G only in a component that is just an edge.

If G has no vertex of degree at least 2, then $\alpha(G) \geq n$; thus G has a component H having a vertex of degree at least 2, and thus every vertex in H has degree at least 2. Assume that G also has an isolated vertex v, and let w be any vertex of H. The degree of w is at least 2, so w has a neighbor z (whose degree is also at least 2). Consider the graph $G' = (G + vw) - wz$; let W' be any independent set in G'. Only one of v and w can be in W', hence $W = (W' \setminus \{v, w\}) \cup \{v\}$ is an independent set in G. So $|W'| \leq |W| \leq n - 1$, thus $\alpha(G') \leq n - 1$. However, in G', vertex v has degree 1, and its neighbor w has degree at least 2, so G' is not α-critical, contradicting the minimality of the number of edges in G. Thus G does not have isolated vertices.

If G has no vertices of degree larger than 2, then every component of G must be an edge or a cycle. There can be no even cycles in G, since we could delete every second edge in an even cycle without increasing the independence number. Thus G must have an odd cycle, and since it has an even number of vertices, it must have at least two odd cycles. Replacing an odd cycle of length k with $k \geq 7$ with an edge and an odd cycle of length $k - 2$ decreases the number of edges, so G can only have 5-cycles. If G had more than two odd cycles, its independence number would be less than $n - 1$, so G must be two 5-cycles and $n - 5$ edges as components. This graph has independence number $n - 1$, and is clearly α-critical.

Now assume that G has a vertex of degree at least 3. Since the number of vertices of odd degree in a graph must be even, there are at least two such vertices. If G contains k edges as components, then the number of edges in G is at least $\frac{1}{2}(2k \cdot 1 + 2 \cdot 3 + 2(2n - 2k - 2)) = 2n - k + 1$. The graph with two 5-cycles and $n - 5$ edges has $n + 5$ edges overall, so we must have $2n - k + 1 \leq n + 5$, and thus $k \geq n - 4$. So G has at least $n - 4$ edges as components, leaving at most eight more vertices; call this subgraph H. A component of H having three or four vertices would be a 4-cycle or contain a triangle, so H is connected and has either six or eight vertices. If H has six vertices, then three neighbors of a vertex of degree at least 3 plus one vertex each from each edge component gives an independent set of size n (since H has no triangles), which is a contradiction. Finally, if H has eight vertices, then in the above computation we must have equality, so there are exactly two vertices of degree 3 in H. Let u be one of these vertices, and v, w, and z be the three vertices adjacent to u. The independence number of H must be 3, so the other four vertices of H must be all adjacent to at least one of these three vertices. Since every degree in H is 2 or 3, one of the vertices among v, w, and z must have degree 3; say z is adjacent to x and y. Each of the remaining two vertices in H must also be adjacent to at least one of v and w, and since their degrees are 2, neither of v or w is adjacent to x or y; so v, w, x, and y form an independent set in H, which is a contradiction.

Thus the only graph satisfying the required properties with a minimum number of edges is the graph with two 5-cycles and $n - 5$ edges as components.

Also solved by George Apostolopoulos (Greece), Con Amore Problem Group (Denmark), Kevin Moss, and the proposer.

Invariant powers of integer parts **December 2011**

1885. *Proposed by H. A. ShahAli, Tehran, Iran.*

Let $n \geq 1$ be an integer and denote by $\lfloor x \rfloor$ the integer part of x. Let $x_1, \ldots, x_n \geq 1$ be real numbers such that

$$\sum_{j=1}^{n} \lfloor x_j^k \rfloor = \sum_{j=1}^{n} \lfloor x_j \rfloor^k \tag{1}$$

is true for infinitely many positive integers k. Prove that all the numbers $x_1, \ldots, x_n$ are integers.

Solution by The Iowa State University Undergraduate Problem Solving Group, Iowa State University, Ames, IA.

For any j, $1 \leq j \leq n$, and any positive integer k, we have that $\lfloor x_j^k \rfloor \geq \lfloor \lfloor x_j \rfloor^k \rfloor = \lfloor x_j \rfloor^k$. Thus, for a given positive integer k, (1) can hold if and only if $\lfloor x_j^k \rfloor = \lfloor x_j \rfloor^k$ for $1 \leq j \leq n$. Now assume that one of the x_j's, say x_1, is not an integer. Then $x_1 = \lfloor x_1 \rfloor + \theta$ for some θ with $0 < \theta < 1$. Let K be a positive integer with $K\theta > 1$. Then for all $k \geq K$,

$$\begin{aligned} \lfloor x_1^k \rfloor &= \lfloor (\lfloor x_1 \rfloor + \theta)^k \rfloor \geq \lfloor \lfloor x_1 \rfloor^k + k\theta \lfloor x_1 \rfloor^{k-1} \rfloor \geq \lfloor \lfloor x_1 \rfloor^k + 1 \rfloor \\ &= \lfloor x_1 \rfloor^k + 1 > \lfloor x_1 \rfloor^k, \end{aligned}$$

where, for the second inequality, we have used the fact that $\lfloor x_1 \rfloor \geq 1$. Thus the inequality in the problem statement cannot hold for infinitely many positive integers k. This completes the proof.

Also solved by George Apostolopoulos (Greece), Elton Bojaxhiu (Germany) and Enkel Hysnelaj (Australia), Robert Calcaterra, Chip Curtis, Dmitry Fleischman, Eugene A. Herman, Joel Iiams, John C. Kieffer, Omran Kouba (Syria), Harris Kwong, Elias Lampakis (Greece), László Lipták, Bob Mallison, Northwestern University Math Problem Solving Group, Joel Schlosberg, Nicholas C. Singer, Marian Tetiva, Alfred Witkowski, Stuart V. Witt, and the proposer.

Answers

Solutions to the Quickies from page 385.

A1021. K is a connected set. Let a and b belong to K with $a < b$. Let $c \in (a, b)$. Suppose that c is not a cluster point of $\{x_n\}$. Let $r = \min\{c - a, b - c\}$. Since c is not a cluster point of $\{x_n\}$, there exist a positive number $\varepsilon < r$ and a positive integer n_0 such that $|x_n - c| \geq \varepsilon$ for all $n \geq n_0$. Since $(x_{n+1} - x_n) \to 0$, there exists $N > n_0$ such that $|x_{n+1} - x_n| < \varepsilon$ for $n > N$. Since a is a cluster point, there exists $m > N$ such that $x_m \leq c - \varepsilon$. If $x_{m+1} \geq c + \varepsilon$, then $x_{m+1} - x_m \geq 2\varepsilon$, which is a contradiction. Given that $|x_{m+1} - c| \geq \varepsilon$, it follows that $x_{m+1} \leq c - \varepsilon$. By induction, this argument shows that $x_k < c - \varepsilon$ for all $k \geq m$, which contradicts the fact that b is a cluster point. Thus c is a cluster point of $\{x_n\}$ and therefore K is connected.

A1022. Both numerator and denominator are linear polynomials in t. Because both of them have the same root $t_0 = \sqrt{1 - 1/a}$, it follows that $f(t)$ is constant for all $t \neq t_0$. Thus $f(t) = f(-1) = 2a + 2\sqrt{a^2 + a} - 1$ for all $t \neq 0$.

REVIEWS

PAUL J. CAMPBELL, *Editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles, books, and other materials are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Rampell, Catherine, 2 from U.S. win Nobel in Economics, *New York Times* (16 October 2012) B1, http://www.nytimes.com/2012/10/16/business/economy/alvin-roth-and-lloyd-shapley-win-nobel-in-economic-science.html .

Economic Sciences Prize Committee of the Royal Swedish Academy of Sciences, Scientific background on the Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012: Stable Allocations and the Practice of Market Design, http://www.nobelprize.org/nobel_prizes/economics/laureates/2012/advanced-economicsciences2012.pdf .

Gale, David, and Lloyd Shapley, College admissions and the stability of marriage, *American Mathematical Monthly* 69 (1962) 9–15.

Roth, Alvin E., Deferred acceptance algorithms: History, theory, practice, and open questions, *International Journal of Game Theory* 36 (2008) (3–4) 537-569. http://dash.harvard.edu/bitstream/handle/1/2579651/Roth_Deferred%20Acceptance.pdf .

Alvin E. Roth (formerly at Harvard, now at Stanford) and Lloyd S. Shapley (UCLA) were awarded the Nobel Prize in Economics “for the theory of stable allocations and the practice of market design.” Shapley is known among mathematicians for the deferred acceptance algorithm in his paper with David Gale; but he is also famous in the arena of social choice for the Shapley-Shubik power index in a voting game. Roth and his students have explored the analysis, creation, and optimization of matching algorithms, including those for medical residents, school choice, kidney allocation, and dorm room assignment (spoiler: your college almost surely is not doing it in a way that is in the best interests of students).

Rathke, Marcie, Independent, negative, canonically Turing arrows of equations and problems in applied formal PDE, *Advances in Pure Mathematics*, http://thatsmathematics.com/blog/wp-content/uploads/2012/09/mathgen-1389529747.pdf .

Eldredge, Nate, Mathgen paper accepted. http://thatsmathematics.com/blog/archives/102.

An online journal, proud of its turnaround time of 10 days for submitted papers, has accepted a randomly-generated mathematics research paper (you too can produce one, at http://thatsmathematics.com/mathgen/) by an author with a name randomly generated from U.S. Census data (shy? you too can get a pseudonym at http://www.kleimo.com/random/name.cfm). The author’s affiliation is the University of Southern North Dakota at Hoople, an invention of Peter Schickele of P.D.Q. Bach fame (spoiler: there is no USND, but there really is a Hoople, ND). And you, too, can be proud to pay the $500 “processing charge” to Scientific Research Publishing to see your paper appear; in fact, you can choose from that organization’s *hundreds* of online journals. At last, mathematics has its own Sokal *Social Choice* affair (Alan D. Sokal, *Beyond the Hoax: Science, Philosophy and Culture* (2010)). (Thanks to Darrah Chavey of Beloit College.)

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Ball, Philip, Proof claimed for deep connection between primes, *Nature News* (10 September 2012) http://www.nature.com/news/proof-claimed-for-deep-connection-between-primes-1.11378 .

Hayes, Brian, The abc game, http://bit-player.org/2012/the-abc-game .

abc conjecture, Wikipedia, http://en.wikipedia.org/wiki/Abc_conjecture .

Nitaj, Abderrahmane, The ABC conjecture home page, http://www.math.unicaen.fr/~nitaj/abc.html .

ABC@home, http://www.abcathome.com/ .

Shinichi Mochizuki (Kyoto University) has released four papers, totaling 500 pages, claiming to have proved the ABC conjecture, based on "inter-universal Teichmüller theory." A similar claim by Lucien Szpiro in 2007 fell through. The conjecture, due to Joseph Oesterlé (U. of Paris) and David W. Masser (U. of Basel), says (roughly) that for coprime a, b, and c, if a or b is divisible by a high power of a prime, then $a + b = c$ must have a large prime factor. For a more precise statement, define the *radical* $\text{rad}(n)$ of an integer n to be the square-free product of the distinct primes dividing n. Then the conjecture says that for any $\varepsilon > 0$, there are only finitely many "exceptional triples" of coprime a, b, c for which $a + b = c$ and $\left(\text{rad}(abc)\right)^{1+\varepsilon} < c$. The intrigue is in the implicit interplay between addition and multiplication. Among easy consequences would be the Thue-Siegel-Roth theorem on diophantine approximation of algebraic numbers, Fermat's Last Theorem for all sufficiently large exponents, the Mordell conjecture, plus various as-yet-unproved results. Hayes casts the conjecture into the form of a game and cites blogs about Mochizuki's papers, while ABC@home lets your computer help search for exceptional triples.

Pitici, Mircea (ed.), *The Best Writing on Mathematics 2012*, Princeton University Press, 2013; xxix + 288 pp, \$19.95(P). ISBN 978-0-691-15655-2.

This is the third annual collection of short pieces of mathematical prose. Like its predecessors, it is oriented toward a general audience and hence is largely equation-free. This year's essays vary from the philosophical ("Why math works," by Mario Livio, and "Is mathematics discovered or invented," by Timothy W. Gowers), to pieces on dance, tessellations, and origami, and essays on mathematics education. Three articles are from the *College Mathematics Journal*, and one from the *American Mathematical Monthly*, but none from this MAGAZINE (maybe next year!). Authors include Terence Tao, Brian Hayes, John C. Baez, and Gerald L. Alexanderson. I recently saw the 2011 volume in a bookstore; so despite coming from a university press, this book may reach some of its target audience.

Gray, Jonathan, Lucy Chambers, and Liliana Bounegru, *The Data Journalism Handbook: How Journalists Can Use Data to Improve the News*, O'Reilly Media, 2012; 250 pp, \$24.99 (P) ISBN 978-1-44933006-4; \$12.99 (ebook: ePub, Mpbi, PDF); free online (piecemeal) at http://datajournalismhandbook.org .

"That journalists need help in math topics [percentage changes, averages] normally covered before high school shows how far newsrooms are from being data literate." This handbook does not treat those topics but instead promotes higher-order skills that would be very helpful to students in courses in data-oriented statistics: finding, formatting, organizing, understanding, and "delivering" data. The book is full of case studies, data visualizations, and tips.

Alsina, Claudi, and Roger B. Nelsen, *Icons of Mathematics: An Exploration of Twenty Key Images*, MAA, 2011; xvii + 327 pp, \$66.95 (MAA member: \$53.95). ISBN 078-0-88385-352-8.

Icon: "a picture that is universally recognized to be representative of something." The authors, who have worked for years on "proofs without words," believe that many such proofs depend on geometric diagrams, such as self-similar figures, tilings, star polygons, Venn diagrams, and right triangles. The icons themselves are points of departure for all kinds of proofs and adventures, including challenge problems for the reader (solutions are given). "All of our icons are two-dimensional; three-dimensional icons will appear in a subsequent work."

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TITLES

PROBLEMS

The letters P, Q, and S refer to Proposals, Quickies, and Solutions,respecitvely; page numbers appear in parentheses. For example,P1888 (62) refers to Proposal 1888, which appears on page 62.

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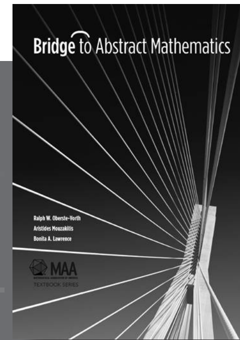

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i. Percent Paid *(15c divided by 15f times 100)*			98%	98%

16. Publication of Statement of Ownership
☒ If the publication is a general publication, publication of this statement is required. Will be printed in the December 2012 issue of this publication.
☐ Publication not required.

17. Signature and Title of Editor, Publisher, Business Manager, or Owner
Beverly Joy Ruedi
Managing Editor

Date
9/5/2012

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